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Promise Date Policies in Inventory Theory

Martin L. Leibowitz and Warren M. Hirsch

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ABSTRACT

The notion of a promise date policy is introduced into an n-period inventory model. The $j^{\underline{th}}$ period consists of the arrival of replenishment stock x_j, receipt of deterministic demand r_j, followed by the issuance of promise dates to all new orders and then the shipment of available stock against outstanding orders promised for the $j^{\underline{th}}$ or earlier periods. The demand schedule $r = (r_1, r_2, \ldots, r_n)$ is associated with an $n \times n$ matrix (d_{uj}) called an urgency structure, $d_{uj} r_j$ being that portion of demand r_j which requires a promise date p such that $j \leq p \leq u$. A $n \times n \times n$ real array is called a promise date policy if and only if the following conditions are satisfied,

$$\sigma_{puj} \geq 0 \ , \ p = 1,\ldots,n \ , \ u = 1,\ldots,n \ , \ j = 1,\ldots,n \ ,$$

$$\sum_{p=1}^{n} \sigma_{puj} = 1 \ , \ u = j,\ldots,n \ , \ j = 1,\ldots,n \ ,$$

$$\sigma_{puj} = 0 \ , \ p > u \text{ or } p < j \ , \ p = 1,\ldots,n \ ,$$
$$u = 1,\ldots,n \ , \ j = 1,\ldots,n \ .$$

The set of all such arrays is denoted by Ω. A policy $(\sigma_{puj}) \in \Omega$ is said to satisfy the FIFO property if promise dates are assigned to orders in the sequence of their receipt. The set of FIFO policies is denoted by $\Omega_F \subset \Omega$. Each $(\sigma_{puj}) \in \Omega$ determines a vector

$r' = (r'_1, r'_2, \ldots, r'_n)$ by the relation,

$$r'_k = \sum_{j=1}^{n} \sum_{u=1}^{n} \sigma_{kuj} d_{uj} r_j \ , \quad k = 1, \ldots, n \ .$$

r' is called a deferred demand schedule and its component r'_k is the total amount of demand promised for $k^{\underline{th}}$ period shipment. The deferred demand schedule r' determines a quantity y_j called the net stock level at the end of the $j^{\underline{th}}$ period,

$$y_j = \sum_{k=1}^{j} x_k - \sum_{k=1}^{j} r'_k \ .$$

The cost of holding stock $y_j \geq 0$ through the end of the $j^{\underline{th}}$ period is $h_j(y_j)$, $y_j \geq 0$. When $y_j < 0$, $(-y_j)$ is the total amount of all unshipped orders promised in the $j^{\underline{th}}$ or earlier periods. The shortage cost in this instance is $h_j(y_j)$, $y_j < 0$. For a fixed replenishment schedule, the total cost is then

$$\sum_{j=1}^{n} h_j(y_j) \ .$$

An optimal promise date policy (σ^*_{puj}) is one which minimizes this cost over all policies in Ω.

The set of all deferred demand schedules corresponding to policies in Ω is denoted by $\mathcal{R}((d_{uj}))$. It is shown that $\mathcal{R}((d_{uj}))$ consists of all n-dimensional vectors r' satisfying the conditions,

$$r'_k \geq 0 \ , \ k = 1,\ldots,n \ ,$$

$$\sum_{k=1}^{n} r'_k = \sum_{j=1}^{n} r_j \ ,$$

$$\sum_{k=1}^{m} r'_k \leq \sum_{j=1}^{m} r_j \ , \ m = 1,\ldots,n \ ,$$

$$\sum_{k=m}^{M} r'_k \geq \sum_{j=m}^{M} r_j \sum_{u=1}^{M} d_{uj} \ , \ 1 \leq m \leq M \leq n \ .$$

The set $\mathcal{R}_F((d_{uj}))$ of deferred demand schedules corresponding to policies in Ω_F is shown to consist of all elements in $\mathcal{R}((d_{uj}))$ satisfying the following additional condition: for each pair of integers (m,M), $1 \leq m \leq M \leq n$, such that

$$\sum_{j=m}^{M} r_j \sum_{u=1}^{M} d_{uj} > 0$$

the following inequality holds,

$$\sum_{k=1}^{M} r'_k \geq \sum_{j=m}^{M} r_j \sum_{u=1}^{M} d_{uj} + \sum_{j=1}^{m-1} r_j \ .$$

The search for optimal policies can then be carried out in the set $\mathcal{R}((d_{uj}))$, or for the FIFO problem, in the set $\mathcal{R}_F((d_{uj}))$. Once an optimal element has been found in $\mathcal{R}((d_{uj}))$ or $\mathcal{R}_F((d_{uj}))$, then appropriate constructions are given for determining a corresponding optimal promise date policy in Ω or Ω_F respectively.

1. INTRODUCTION AND SUMMARY

Inventory theory is concerned primarily with procedures for minimizing the cost of replenishing, holding and being without a stock of needed goods. In most models, including the one considered in this paper, the problem is viewed in terms of a finite number of sequential time periods. Within each of these periods, the basic inventory cycle of replenishment, receipt of orders, and shipment of stock against current and earlier orders takes place. Three components of cost are incurred in the course of each period. The replenishment cost is assumed usually to be a convex function of the amount of stock received, indicating the possibility of greater economy through increased volume. The cost of holding stock is assumed to be a function of the remaining stock level after replenishment and shipment. The shortage cost--the cost of not being able to fill orders on time--is assumed usually to depend on the total amount of all unshipped orders at the end of the period.

The determination of when an order becomes late is the point of departure for the problems considered in this study. In the literature on mathematical inventory theory, it is assumed almost invariably that any order not shipped during its receipt period becomes late and must be included in evaluating the shortage cost. We take the alternative view that some orders received in a given period are more urgent than others. We formalize this viewpoint by assuming that

each order received in the j^{th} period has an explicit urgency deadline u, $j \leq u \leq n$. We then introduce the issuance of promise dates as a new event following receipt of orders within each period's cycle.(1) A promise date p, $j \leq p \leq u$, represents the supplier's promise to the customer to ship his order in the p^{th} period. We treat an order as late and include it in the computation of the shortage cost if and only if it is not shipped on its promise date.

For mathematical simplicity both the replenishment and the demand are assumed to be deterministic, known quantities.(2)

A procedure for assigning promise dates to orders will be referred to as a promise date policy. For a given replenishment sequence, an optimal promise date policy is one which minimizes the total holding and shortage costs over all periods considered.

We develop in Section 2.1 the basic mathematical model of a promise date problem under the assumption that all orders have the n^{th} period as their urgency deadline. This simplifying assumption is helpful in formulating the initial model, and leads to several theorems that are needed later in treating the more general problem. A set $\mathcal{A}$ of $n \times n$ real "deferral"

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(1) This is a conventional practice in many industries.

(2) A more realistic model in which these quantities are treated as random variables will be discussed in a subsequent publication.

matrices is shown to be equivalent to the class of all feasible promise date policies. Each matrix $(a_{pj}) \in \mathcal{A}$ determines a vector r' which we refer to as a deferred demand schedule. This relation induces equivalence classes in $\mathcal{A}$ consisting of matrices corresponding to a single r'. The overall cost is shown to be constant on each such equivalence class. Consequently, the problem reduces to first finding an optimal element r'^* in the set $\mathcal{R}$ of all possible deferred demand schedules, and then determining any matrix $(a^*_{pj}) \in \mathcal{A}$ corresponding to r'^*.

In Section 2 we single out the subset $\mathcal{A}_F \subset \mathcal{A}$ of matrices having the FIFO (First In, First Out) property of issuing promise dates to orders in the sequence of their receipt, and we describe a method for constructing such matrices.

In Section 2.3, we obtain a useful characterization of the set $\mathcal{R}$, and show how to apply the preceding construction to the elements of this set. It is shown that this construction leads to an essentially unique element (a_{pj}) in $\mathcal{A}_F$ corresponding to a given r'.

In Section 2.4, a method for finding a unique optimal $r'^* \in \mathcal{R}$ is developed.

Section 3.1 introduces the urgency structure matrix (d_{uj}), the components d_{uj} being the fraction of $j^{\underline{th}}$ period demand having a $u^{\underline{th}}$ period urgency deadline. A promise date policy under an urgency structure (d_{uj}) consists of an $n \times n \times n$ real array (σ_{puj}). The component σ_{puj} is the fraction of $j^{\underline{th}}$ period

demand having a $u^{\underline{th}}$ period urgency deadline and a $p^{\underline{th}}$ period promise date. The problem is to select an optimal policy (σ^*_{puj}) from the set Ω of all feasible promise date policies.

Section 3.2 shows that each $(\sigma_{puj}) \in \Omega$ gives rise to a deferral matrix $(a_{kj}) \in \mathcal{A}$ bearing the same relation to a deferred demand vector r' as in Section 2. Moreover, the cost of the policy (σ_{puj}) is shown to depend only on this associated deferred demand vector r'. A characterization is then developed for the set $\mathcal{A}((d_{uj})) \subseteq \mathcal{A}$ of deferral matrices arising from policies in Ω. We demonstrate also a method for determining a policy $(\sigma_{puj}) \in \Omega$ associated with a given deferral matrix $(a_{kj}) \in \mathcal{A}((d_{uj}))$.

In Section 3.3, we focus attention on the special case when only promise date policies having the FIFO property are of interest. A characterization is obtained for the set $\mathcal{K}_F((d_{uj}))$ of all deferred demand vectors corresponding to such policies. Once an optimal element $r'^* \in \mathcal{K}_F((d_{uj}))$ is found by some optimization technique, a construction is given for determining an associated optimal FIFO policy (σ^*_{puj}). An example shows that such a policy need not be optimal when the FIFO property is not required.

Finally, in Section 3.4, a characterization is obtained for the set $\mathcal{K}((d_{uj}))$ of all deferred demand vectors associated with feasible promise date policies. A constructive technique is provided for determining an optimal promise date policy $(\sigma^*_{puj}) \in \Omega$ associated with an optimal element $r'^* \in \mathcal{K}((d_{uj}))$.

The problems, concepts, and results described in this paper represent only a first small step toward the development of a comprehensive theory of promise date policies for inventory management. It is hoped that the formulations introduced here will lead ultimately to the development of models that realistically reflect all the significant features found in operational situations.

2. THE SIMPLEST DEFERRAL PROBLEM: UNSTRUCTURED, DETERMINISTIC DEMAND AND FIXED REPLENISHMENT SCHEDULE

2.1 Basic Model

The supplier's inventory situation is viewed as a repeated cycle of events taking place within each of n discrete periods. The first event in the $j^{\underline{th}}$ period is the receipt of replenishment stock, x_j,

$$x_j \geq 0 , \quad j = 1,\ldots,n . \tag{2.1.1}$$

The second event is the receipt of orders. The total amount ordered in the $j^{\underline{th}}$ period is represented by a real number r_j satisfying the inequality

$$r_j \geq 0 , \quad j = 1,\ldots,n . \tag{2.1.2}$$

It will be assumed that the total demand and stock replenishment are balanced over the n periods, i.e.,

$$\sum_{j=1}^{n} r_j = \sum_{j=1}^{n} x_j . \tag{2.1.3}$$

This assumption greatly simplifies the notation and proof.

The demand is considered to consist of orders that can wait until the $n^{\underline{th}}$ period for shipment. In other words, all customers are totally patient, i.e., orders have the $n^{\underline{th}}$ period as their urgency deadline.

The third event within the $j^{\underline{th}}$ period is the issuance of promise dates to all orders comprising the

demand r_j. (All promise dates discussed in these thesis are promised dates of shipment. Delivery promises can readily be generated from these promise dates by inclusion of an appropriate delivery delay for each order.)
The assignment of promise dates is described by quantities,

$$0 \leq a_{pj} \leq 1 , \quad p = 1,\ldots,n ,$$

where a_{pj} is that fraction of demand r_j to be provided with a promised shipment date of the $p^{\underline{th}}$ period. The quantities,

$$a_{pj} , \quad p = 1,\ldots,n , \quad j = 1,\ldots,n ,$$

form a matrix which shall be referred to as a "deferral matrix." The total of all orders promised for the $p^{\underline{th}}$ period shipment is then represented by r'_p, where

$$(2.1.4) \qquad r'_p = \sum_{j=1}^{n} a_{pj} r_j , \quad p = 1,\ldots,n .$$

The vector $r' = (r'_1,\ldots,r'_n)$ will be called the "deferred demand schedule."

The fourth and last event of the $j^{\underline{th}}$ period consists of the shipment of available stock against outstanding orders promised for the $j^{\underline{th}}$ or earlier periods. Define the "net stock level" y_j at the end of the $j^{\underline{th}}$ period to be

$$(2.1.5) \qquad y_j = \sum_{k=1}^{j} x_k - \sum_{k=1}^{j} r'_k , \quad j = 1,\ldots,n .$$

When

$$y_j \geq 0 ,$$

this quantity represents the stock remaining at the end of the $j^{\underline{th}}$ period. When

$$y_j < 0 ,$$

the quantity $-y_j$ represents orders promised by the $j^{\underline{th}}$ period but unshipped.

During this cycle of events three types of costs are charged against the supplier. The first is the cost of the replenishment stock x_j,

$$c_j(x_j) .$$

The second cost, which is incurred when $y_j \geq 0$, results from carrying an amount y_j of stock through the end of the $j^{\underline{th}}$ period. This is referred to as the "holding cost" and is denoted by

$$h_j(y_j) , \quad y_j \geq 0 .$$

The third cost, which is incurred when $y_j < 0$, is the customer disappointment, loss of goodwill, and potential loss of future business that arises as a consequence of shipping orders after their promise dates. This shortage (or "late-on-promise") cost will be represented by

$$h_j(y_j) , \quad y_j < 0 .$$

The overall system cost for the n periods then becomes

$$(2.1.6) \qquad \sum_{j=1}^{n} c_j(x_j) + \sum_{j=1}^{n} h_j(y_j) \ .$$

In this model, the supplier's sole decision is to select a deferral matrix. Clearly, this matrix must satisfy certain obvious contraints. This suggests the following definition.

Definition 2.1.1. An $n \times n$ real matrix (a_{pj}) is said to be a "feasible deferral matrix" if it satisfies the three conditions,

$$(2.1.7) \qquad \sum_{p=1}^{n} a_{pj} = 1 \ , \quad j = 1,\ldots,n \ ,$$

$$(2.1.8) \qquad a_{pj} \geq 0, \quad p = 1,\ldots,n \ , \ j = 1,\ldots,n \ ,$$

$$(2.1.9) \qquad a_{pj} = 0 \ , \ p < j \ , \ p = 1,\ldots,n \ , \ j = 1,\ldots,n \ .$$

The set of all feasible deferral matrices will be denoted by $\mathcal{A}$.

The total cost is then a function

$$G_0 : \mathcal{A} \rightarrow \mathcal{R} \ ,$$

where

$$(2.1.10) \qquad G_0((a_{pj})) = \sum_{j=1}^{n} c_j(x_j) + \sum_{j=1}^{n} h_j(y_j) \ .$$

An optimal policy is one which minimizes G_0. Moreover, since the replenishment schedule is fixed, the optimization

problem reduces to finding a deferral policy (a_{pj}) minimizing

$$\sum_{j=1}^{n} h_j(y_j)$$

among all feasible policies in $\mathcal{A}$.

Two deferral matrices are defined to be equivalent if they give rise to the same deferred demand schedule. Each equivalence class induced in $\mathcal{A}$ by this relation corresponds to a single feasible deferred demand schedule, which leads to the following definition.

Definition 2.1.2. An n-dimensional real vector r' is said to be a "feasible deferred demand schedule" (associated with a given demand schedule r) if there exists a feasible deferral matrix (a_{pj}) such that

$$(2.1.11) \qquad r'_p = \sum_{j=1}^{n} a_{pj} r_j \ , \quad p = 1,..,n \ .$$

We denote by $\mathcal{R}$ the set of all feasible deferred demand schedules associated with a given demand schedule r. Although $\mathcal{R}$ depends on r, we do not explicitly display this dependence in the notation.

It is plain that the function G_0 is constant on each equivalence class in $\mathcal{A}$. Consequently, G_0 can be regarded as a function on the set $\mathcal{R}$ of feasible deferred demand schedules. The problem then reduces to first choosing an optimal schedule r'^* from the set $\mathcal{R}$, and

then constructing any element (a^*_{pj}) in the equivalence class corresponding to r'*.

2.2. The FIFO Construction

In some applications management may insist that promise date be assigned on a "First-In, First-Out" (FIFO) basis, i.e.,

$$\max\{p: a_{pj} > 0\} \leq \min\{p: a_{pk} > 0\} , \tag{2.2.1}$$

$$r_j > 0 , \; r_k > 0 ,$$

$$k = j+1,\ldots,n , \; j = 1,\ldots,n-1 .$$

The class of FIFO policies $\mathcal{A}_F$ is therefore the subset of feasible policies $\mathcal{A}$ whose elements satisfy (2.2.1).

We propose a constructive process for the determination of a feasible deferral matrix satisfying (2.2.1). This procedure is defined by the recursion formula, for $r_j = 0$,

$$a_{pj} = \begin{cases} 0 & p = 1,\ldots,j-1 \\ 1 & p = j \\ 0 & p = j+1,\ldots,n , \end{cases} \tag{2.2.2a}$$

and for $r_j > 0$,

$$(2.2.2b)\qquad a_{pj} = \begin{cases} 0\ , & p = 1,\ldots,j-1 \\ \dfrac{1}{r_j}\left(r'_j - \sum_{k=1}^{j-1} a_{jk} r_k\right)\ , & p = j \\ \min\left\{\dfrac{1}{r_j}\left(r'_p - \sum_{k=1}^{j-1} a_{pk} r_k\right), 1 - \sum_{t=1}^{p-1} a_{tj}\right\}, & \\ \quad p = j+1,\ldots,n\ , & \end{cases}$$

with the recursion beginning with $p = 1$, $j = 1$, stepping through the p-index until $p = n$, $j = 1$, beginning again with $p = 1$, $j = 2$, and proceeding through $p = n$, $j = 2$, etc., until terminating at $p = n$, $j = n$.

We have yet to characterize the set $\mathcal{R}$ and without this characterization, we cannot prove that the matrix generated by (2.2.2) satisfies (2.1.7), (2.1.8), (2.1.9), and (2.2.1). The mathematical formulation and the proofs of these claims, together with the dual optimization problems which they imply, comprise the content of the next section. Before turning to these formal proofs, we give an example to illustrate the construction.

Example 2.2.1: Illustration of FIFO Construction.

For $n = 3$, we are given

$$r_1 = 4 \qquad r_2 = 1 \qquad r_3 = 7$$

and

$$r'_1 = 2 \qquad r'_2 = 1 \qquad r'_3 = 9\ .$$

Applying (2.2.2) for $p = 1$, $j = 1$, we obtain

$$a_{11} = \frac{1}{r_1}(r_1' - \sum_{k=1}^{0} a_{1k}r_k) = \frac{r_1'}{r_1} = \frac{2}{4} = \frac{1}{2} .$$

Then, for $p = 2$, $j = 1$,

$$a_{21} = \min\{\frac{r_2'}{r_1}, 1 - a_{11}\} = \min\{\frac{1}{4}, 1 - \frac{1}{2}\} = \frac{1}{4} ,$$

and for $p = 3$, $j = 1$,

$$a_{31} = \min\{\frac{r_3'}{r_1}, 1 - a_{11} - a_{21}\} = \min\{\frac{9}{2}, 1 - \frac{1}{2} - \frac{1}{4}\} = \frac{1}{4} .$$

Moving next to $p = 1$, $j = 2$, we immediately have

$$a_{12} = 0 .$$

Then, for $p = 2$, $j = 2$,

$$a_{22} = \frac{1}{r_2}(r_2' - \sum_{k=1}^{1} a_{2k}r_k) = \frac{1}{r_2}(r_2' - a_{21}r_1)$$

$$= \frac{1}{1}(1 - \frac{4}{4}) = 0 ,$$

and for $p = 3$, $j = 2$,

$$a_{32} = \min\{\frac{1}{r_2}(r_3' - a_{31}r_1), 1 - a_{12} - a_{22}\}$$

$$= \min\{\frac{1}{1}(9 - \frac{4}{4}), 1 - 0 - 0\} = 1 .$$

For $j = 3$, we immediately obtain

$$a_{13} = a_{23} = 0 ,$$

and

$$a_{33} = \frac{1}{r_3}(r_3' - \sum_{k=1}^{2} a_{3k}r_k) = \frac{1}{7}(9 - \frac{1}{4}\cdot 4 - 1\cdot 1) = 1 .$$

Thus, the FIFO construction yields the matrix

$$(a_{pj}) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 1 & 1 \end{pmatrix} .$$

This matrix clearly fulfills the requirements (2.1.7), (2.1.8), and (2.1.9) for a feasible deferral matrix. Moreover, it also has the property (2.2.1), thereby qualifying it as a FIFO policy.

2.3. Characterization of the Set $\mathcal{R}$ and Its Role as a Solution Space

In Section 2.1 the p^{th} component of the deferred demand schedule r' was defined formally by (2.1.4),

$$r'_p = \sum_{j=1}^{n} a_{pj} r_j , \quad p = 1,\ldots,n ,$$

where (a_{pj}) is the feasible deferral matrix selected by the supplier. Clearly, not all n-dimensional real vectors qualify as feasible deferred demand schedules for a given demand schedule r. Certain consistency conditions are needed. From the viewpoint of the model interpretation, non-negativity is one such condition,

(2.3.1) $$r'_p \geq 0 , \quad p = 1,\ldots,n .$$

A second condition, "complete deferral," arises from the model's requirement that every incoming order be given some promise date,

(2.3.2) $$\sum_{p=1}^{n} r'_p = \sum_{j=1}^{n} r_j .$$

A third requirement of the model is that an order can never be promised for shipment in a period prior to its arrival period. This implies that the sum of all orders promised for shipment for the $m^{\underline{th}}$ period can never exceed the sum of all orders received by the $m^{\underline{th}}$ period. This is expressed in the relation

(2.3.3) $$\sum_{p=1}^{m} r'_p \leq \sum_{j=1}^{m} r_j , \quad m = 1,\ldots,n .$$

As will be proved in the next theorem, these three properties are necessary and sufficient conditions for a vector r' to be a feasible deferred demand schedule. These three properties then provide a characterization of the set $\mathcal{R}$. It will also be shown that the problem of finding an optimal deferral policy (a^*_{pj}) can be reduced to a much simpler optimization problem in the set $\mathcal{R}$.

Theorem 2.3.1. (Characterization of the Set $\mathcal{R}$). Let r be a demand schedule and let $\mathcal{R}$ be the associated set of feasible deferred demand schedules. Then $\mathcal{R}$ consists

of all n-dimensional real vectors satisfying (2.3.1), (2.3.2), and (2.3.3).

Proof: The theorem will be proved if we can show that the conditions (2.3.1), (2.3.2), and (2.3.3) are necessary and sufficient for a n-dimensional real vector to belong to $\mathcal{R}$.

We will first prove that these three conditions are necessary. Let r' be an element in $\mathcal{R}$.

1) Non-negativity of r'_p (2.3.1).

Obvious from (2.1.2), (2.1.8), and the defining relation (2.1.4).

2) Complete Deferral (2.3.2).

Summing (2.1.4) over p,

$$\sum_{p=1}^{n} r'_p = \sum_{p=1}^{n} \sum_{j=1}^{n} a_{pj} r_j = \sum_{j=1}^{n} r_j \left(\sum_{p=1}^{n} a_{pj}\right) = \sum_{j=1}^{n} r_j ,$$

where we have used (2.1.7) to get the needed result.

3) Arrival Consistency (2.3.3).

From (2.1.7) and (2.1.8),

$$\sum_{p=1}^{m} a_{pj} \leq 1 , \quad m = 1,\ldots,n , \quad j = 1,\ldots,n .$$

Then, from (2.1.9) and (2.1.4), for p = 1,...,n,

$$r'_p = \sum_{j=1}^{n} a_{pj} r_j = \sum_{j=1}^{p} a_{pj} r_j ,$$

or for $p \leq m$,

$$r'_p = \sum_{j=1}^{p} a_{pj} r_j + \sum_{j=p+1}^{m} a_{pj} r_j = \sum_{j=1}^{m} a_{pj} r_j \ , \ p = 1,\ldots,m \ .$$

Combining these two results,

$$\sum_{p=1}^{m} r'_p = \sum_{p=1}^{m} \sum_{j=1}^{m} a_{pj} r_j = \sum_{j=1}^{m} r_j \left(\sum_{i=1}^{m} a_{pj}\right) \leq \sum_{j=1}^{m} r_j \ ,$$

$$m = 1,\ldots,n \ .$$

This completes the proof of necessity.

To prove sufficiency, let r' be any n-dimensional real vector satisfying (2.3.1), (2.3.2), and (2.3.3). From Definition 2.1.2, we see that we must demonstrate the existence of a matrix (a_{pj}) in $\mathcal{A}$ which transforms r into r' via the relation (2.1.4). The underlying idea of the following proof is to apply the FIFO construction to the vectors r and r', and to show that the resulting matrix is an element of $\mathcal{A}$ and relates to r' via (2.1.4).

Lemma 2.3.1. Let r' be an n-dimensional real vector satisfying (2.3.1), (2.3.2), and (2.3.3), and let (a_{pj}) be the matrix determined by the recursion (2.2.2) applied to r and r'. Then,

$$(2.3.4) \quad r'_p \geq \sum_{k=1}^{j} a_{pk} r_k \ , \ p = 1,\ldots,n \ , \ j = 1,\ldots,n \ .$$

Proof: The proof proceeds by induction on the j-index.

For $j = 1$ and $r_1 > 0$, the FIFO construction (2.2.2) yields,

$$a_{p1} = \begin{cases} \dfrac{r'_1}{r_1} & , \ p = 1 \\ \min\{\dfrac{r'_p}{r_1}, 1 - \sum_{t=1}^{p-1} a_{t1}\} & , \ p = 2,\ldots,n \end{cases}$$

$$\leq \frac{r'_p}{r_1} \quad , \ p = 1,\ldots,n \ ,$$

which is the needed result. For $j = 1$ and $r_1 = 0$, we have from (2.2.2),

$$a_{p1} = \begin{cases} 1 \ , & p = 1 \\ 0 \ , & p = 2,\ldots,n \ , \end{cases}$$

and from (2.3.1) and (2.3.3),

$$r'_1 = 0 = a_{11} r_1$$

and

$$r'_p \geq 0 = a_{p1} r_1 \ , \quad p = 2,\ldots,n \ ,$$

which immediately gives us (2.3.4) for $j = 1$.

Now assume the lemma holds for

$$j = 1,\ldots,(\hat{j}-1) \ , \quad 1 < \hat{j} \leq n \ .$$

First we consider $r_{\hat{j}} > 0$. For the $\hat{j}^{\text{th}}$ stage, the FIFO construction (2.2.2) yields

$$a_{p\hat{j}} = \begin{cases} 0 & , \ p = 1,\ldots,\hat{j}-1 \\ \frac{1}{r_{\hat{j}}}(r'_{\hat{j}} - \sum_{k=1}^{\hat{j}-1} a_{\hat{j}k} r_k) & , \ p = \hat{j} \\ \min\{\frac{1}{r_{\hat{j}}}(r'_p - \sum_{k=1}^{\hat{j}-1} a_{pk} r_k), 1 - \sum_{t=1}^{p-1} a_{t\hat{j}}\} & , \ p = \hat{j}+1,\ldots,n \ , \end{cases}$$

and so we have

$$a_{p\hat{j}} \leq \frac{1}{r_{\hat{j}}}(r'_p - \sum_{k=1}^{\hat{j}-1} a_{pk} r_k) \ , \quad p = \hat{j},\ldots,n \ ,$$

which is just (2.3.4). For lower p-values, we have, using (2.1.9),

$$\sum_{k=1}^{\hat{j}} a_{pk} r_k = a_{p\hat{j}} r_{\hat{j}} + \sum_{k=1}^{\hat{j}-1} a_{pk} r_k = \sum_{k=1}^{\hat{j}-1} a_{pk} r_k$$

$$\leq r'_p \ , \quad p = 1,\ldots,\hat{j}-1 \ ,$$

where the last inequality follows from the induction hypothesis.

Finally, for the case of

$$r_{\hat{j}} = 0 \ ,$$

we have

$$a_{p\hat{j}} = \begin{cases} 0 \ , & p = 1,\ldots,\hat{j}-1 \\ 1 \ , & p = \hat{j} \\ 0 \ , & p = \hat{j}+1,\ldots,n \ , \end{cases}$$

and, as before,

$$\sum_{k=1}^{\hat{j}} a_{pk} r_k = \sum_{k=1}^{\hat{j}-1} a_{pk} r_k \leq r'_p \ , \ p = 1, \ldots, \hat{j} \ ,$$

by the induction hypothesis.

This completes the induction and the lemma is proved.

Lemma 2.3.2. Let r' be an n-dimensional real vector satisfying (2.3.1), (2.3.2), and (2.3.3), and let (a_{pj}) be the matrix determined by the recursion (2.2.2) applied to r and r'. Then there exists at least one integer L(1),

$$1 \leq L(1) \leq n \ ,$$

such that

$$r'_{L(1)} \geq r_1 \left(1 - \sum_{t=1}^{L(1)-1} a_{t1}\right) \ .$$

Moreover, letting $\hat{L}(1)$ be the smallest of these integers, then the quantities a_{p1} satisfy the following relations, for $r_1 = 0$,

$$a_{p1} = \begin{cases} 1 \ , & p = 1 \\ \\ 0 \ , & p = 2, \ldots, n \ , \end{cases}$$

and for $r_1 > 0$,

$$a_{p1} = \begin{cases} \dfrac{r'_p}{r_1} & , \ p = 1, \ldots, \hat{L}(1)-1 \\ 1 - \sum_{t=1}^{\hat{L}(1)-1} a_{t1} & , \ p = \hat{L}(1) \\ 0 & , \ p = \hat{L}(1)+1, \ldots, n \ . \end{cases}$$

Proof: First, we must demonstrate the existence of an integer L(1).

For $r_1 = 0$, each p-value is an L(1) by virtue of (2.3.1).

For $r_1 > 0$, suppose that none of the p-values in the range $p = 1,\ldots,n-1$ qualifies as an L(1), i.e.,

$$r'_p < r_1(1 - \sum_{t=1}^{p-1} a_{t1}) , \quad p = 1,\ldots,n-1 .$$

Then, the FIFO construction (2.2.2) must have the form,

$$a_{p1} = \begin{cases} \frac{r'_1}{r_1} & , \quad p = 1 \\ \min\{\frac{r'_p}{r_1}, 1 - \sum_{t=1}^{p-1} a_{t1}\} & , \quad p = 2,\ldots,n \end{cases}$$

$$= \begin{cases} \frac{r'_p}{r_1} & , \quad p = 1,\ldots,n-1 \\ \min\{\frac{r'_n}{r_1}, 1 - \sum_{t=1}^{n-1} a_{t1}\} & , \quad p = n , \end{cases}$$

so that

$$r_1(1 - \sum_{t=1}^{n-1} a_{t1}) = r_1 - \sum_{t=1}^{n-1} (\frac{r'_t}{r_1})r_1 = r_1 - \sum_{t=1}^{n-1} r'_t .$$

But from (2.3.2) and (2.1.2),

$$\sum_{t=1}^{n} r'_t \geq r_1$$

or

$$r'_n \geq r_1 - \sum_{t=1}^{n-1} r'_t = r_1\left(1 - \sum_{t=1}^{n-1} a_{t1}\right) ,$$

thereby proving that $p = n$ qualifies as an $L(1)$ value if no earlier p-value does. Consequently, at least one $L(1)$ integer must exist.

Now let $\hat{L}(1)$ be the smallest such integer. Then, for $r_1 > 0$, the FIFO construction (2.2.2) immediately yields

$$a_{p1} = \frac{r'_p}{r_1} , \quad p = 1,\ldots,\hat{L}(1)-1 ,$$

since by definition of $\hat{L}(1)$,

$$\frac{r'_p}{r_1} < 1 - \sum_{t=1}^{p-1} a_{t1} , \quad p = 1,\ldots,\hat{L}(1)-1 .$$

For $p = \hat{L}(1)$, on the other hand, we have

$$r'_{\hat{L}(1)} \geq r_1\left(1 - \sum_{t=1}^{\hat{L}(1)-1} a_{t1}\right)$$

and the FIFO construction generates

$$a_{\hat{L}(1),1} = 1 - \sum_{t=1}^{\hat{L}(1)-1} a_{t1}$$

as needed. Finally, for $p = \hat{L}(1)+1$, we have

$$0 = 1 - \sum_{t=1}^{\hat{L}(1)} a_{t1} ,$$

so that the FIFO construction becomes

$$a_{\hat{L}(1)+1,1} = \min\{r_{\hat{L}(1)+1}/r_1, 0\} = 0$$

because of the non-negativity of $r'_{\hat{L}(1)+1}$ and r_1. However, we then have

$$0 = 1 - \sum_{t=1}^{\hat{L}(1)+1} a_{t1} .$$

By a finite number of repetitions of the above argument we obtain

$$a_{p1} = 0 , \quad p = \hat{L}(1)+1, \ldots, n .$$

This shows that for $r_1 > 0$, the elements

$$a_{p1} , \quad p = 1, \ldots, n ,$$

have the stated representation.

For $r_1 = 0$, the representation stated by the lemma follows immediately by definition.

This completes the proof of the lemma.

Lemma 2.3.3. Let r' be an n-dimensional real vector satisfying (2.3.1), (2.3.2), and (2.3.3), and let (a_{pj}) be the matrix determined by the recursion (2.2.2) applied to r and r'. Suppose, for some fixed j, there exists numbers L(j),

$$j \leq L(j) \leq n , \tag{2.3.5}$$

satisfying the relation,

$$(2.3.6)\quad r'_{L(j)} - \sum_{k=1}^{j-1} a_{L(j),k} r_k \geq r_j \left(1 - \sum_{t=1}^{L(j)-1} a_{tj}\right) .$$

Let $\hat{L}(j)$ be the smallest such number. Suppose further the matrix components,

$$a_{pj} , \quad p = 1,\ldots,n ,$$

satisfy the following relations, for $r_j = 0$,

$$(2.3.7a)\qquad a_{pj} = \begin{cases} 0 , & p = 1,\ldots,j-1 \\ 1 , & p = j \\ 0 , & p = j+1,\ldots,n , \end{cases}$$

and for $r_j > 0$

$$(2.3.7b)\quad a_{pj} = \begin{cases} 0 & , \ p = 1,\ldots,j-1 \\ \frac{1}{r_j}\left(r'_p - \sum_{k=1}^{j-1} a_{pk} r_k\right) & , \ p = j,\ldots,\hat{L}(j)-1 \\ 1 - \sum_{t=1}^{p=1} a_{tj} & , \ p = \hat{L}(j) \\ 0 & , \ p = \hat{L}(j)+1,\ldots,n . \end{cases}$$

Then these matrix components must have the following properties:

1) non-negativity

$$a_{pj} \geq 0 , \quad p = 1,\ldots n ,$$

2) conservation

$$\sum_{p=1}^{n} a_{pj} = 1 ,$$

3) arrival limit

$$a_{pj} = 0 \ , \ p = 1,\ldots,j-1 \ .$$

Proof: The lemma is trivial when

$$r_j = 0 \ .$$

The remainder of the proof assumes $r_j > 0$.

The lemma hypothesizes the structure (2.3.7) for

$$a_{pj} \ , \quad p = 1,\ldots,n \ .$$

From this structure and Lemma 2.3.1, we have

$$a_{pj} = (r'_p - \sum^{j-1} a_{pk} r_k)/r_j \geq 0 \ , \ p = j,\ldots,\hat{L}(j)-1 \ .$$

For $p = \hat{L}(j)$, we must consider three distinct cases. First, we consider the case,

$$\hat{L}(j) > j + 1 \ ,$$

then the FIFO construction (2.2.2) insures that

$$a_{\hat{L}(j)-1,j} \leq 1 - \sum_{t=1}^{\hat{L}(j)-2} a_{tj} \ ,$$

so that

$$a_{\hat{L}(j),j} = 1 - \sum_{t=1}^{\hat{L}(j)-1} a_{tj} \geq 0 \ .$$

For the case,

$$\hat{L}(j) = j + 1 \ ,$$

the definition of $\hat{L}(j)$ provides

$$r'_j - \sum_{k=1}^{j-1} a_{jk} r_k < r_j(1 - \sum_{t=1}^{j-1} a_{tj}) = r_j \ .$$

Then, from (2.3.7),

$$a_{jj} = \frac{1}{r_j}(r'_j - \sum_{k=1}^{j-1} a_{jk} r_k) < 1 \ ,$$

and hence

$$a_{\hat{L}(j),j} = a_{j+1,j} = 1 - a_{jj} > 0 \ .$$

For the third possibility,

$$\hat{L}(j) = j \ ,$$

we immediately have from (2.3.7), that

$$a_{\hat{L}(j),j} = a_{jj} = 1 \geq 0 \ .$$

Since (2.3.7) insures that

$$a_{pj} = 0 \ , \quad p = 1,\ldots,j-1$$

and

$$a_{pj} = 0 \ , \quad p = \hat{L}(j)+1,\ldots,n \ ,$$

we have proved that

$$a_{pj} \geq 0 \ , \quad p = 1,\ldots,n \ .$$

The conservation property follows directly from

the structure (2.3.7),

$$\sum_{p=1}^{n} a_{pj} = \sum_{p=1}^{\hat{L}(j)} a_{pj} = a_{\hat{L}(j),j} + \sum_{p=1}^{\hat{L}(j)-1} a_{pj}$$

$$= (1 - \sum_{p=1}^{\hat{L}(j)-1} a_{pj}) + \sum_{p=1}^{\hat{L}(j)-1} a_{pj}$$

$$= 1 .$$

The "arrival limit" property is an immediate consequence of the hypothesized structure (2.3.7).

Lemma 2.3.4. Let r' be an n-dimensional real vector satisfying (2.3.1), (2.3.2), and (2.3.3), and let (a_{pj}) be the matrix determined by the recursion (2.2.2) applied to r and r'. Suppose that for all j-values in the range,

$$j = 1,\ldots,(J-1) , \quad 1 < J \leq n ,$$

there exist numbers L(j),

$$j \leq L(j) \leq n$$

satisfying the relation (2.3.6),

$$r'_{L(j)} - \sum_{k=1}^{j-1} a_{L(j),k} r_k \geq r_j (1 - \sum_{t=1}^{L(j)-1} a_{tj}) .$$

Moreover, let $\hat{L}(j)$ be the smallest such number for each j, and further suppose that the matrix components

$$a_{pj} , \quad p = 1,\ldots,n , \quad j = 1,\ldots,J-1 ,$$

satisfy the relation (2.3.7). Then there exists an integer $\hat{L}(J)$ and the components,

$$a_{pJ} , \quad p = 1,\ldots,n ,$$

also satisfy (2.3.7) relative to this $\hat{L}(J)$.

Proof: First, we must show that there exists an integer $L(J)$ in the range

$$J \leq L(J) \leq n ,$$

satisfying (2.3.6).

We proceed by assuming the contrary, i.e., that (2.3.6) is not fulfilled by any integer in the specified range. This contrary supposition then implies the following strict inequalities

$$(2.3.8) \qquad r'_p - \sum_{k=1}^{J-1} a_{pk} r_k < r_J \left(1 - \sum_{t=1}^{p-1} a_{tJ}\right) ,$$

$$p = J,\ldots,n .$$

Consider first the case of $r_J > 0$. The inequalities (2.3.8) insure that the FIFO construction (2.2.2) will yield

$$a_{pJ} = \begin{cases} 0 & , \ p = 1,\ldots,J-1 \\ \frac{1}{r_J}\left(r'_p - \sum_{k=1}^{J-1} a_{pk} r_k\right) & , \ p = J,\ldots,n . \end{cases}$$

Summing over these components,

$$\sum_{t=1}^{p-1} a_{tJ} = \begin{cases} 0 & , \ p = 1,\ldots,J \\ \frac{1}{r_J} \sum_{t=J}^{p-1} (r'_t - \sum_{k=1}^{J-1} a_{tk} r_k) & , \ p = J+1,\ldots,n \end{cases}$$

$$= \frac{1}{r_J} \sum_{t=J}^{p-1} (r'_t - \sum_{k=1}^{J-1} a_{tk} r_k) \ , \ p = 1,\ldots,n \ .$$

Inserting this expression into the inequality (2.3.8) implied by our contrary supposition,

$$r'_p - \sum_{k=1}^{J-1} a_{pk} r_k < r_J - \sum_{t=J}^{p-1} (r'_t - \sum_{k=1}^{J-1} a_{tk} r_k) \ ,$$

$$\sum_{t=J}^{p} r'_t < r_J + \sum_{k=1}^{J-1} a_{pk} r_k + \sum_{t=J}^{p-1} \sum_{k=1}^{J-1} a_{tk} r_k \ ,$$

and

$$\sum_{t=J}^{p} r'_t < r_J + \sum_{t=J}^{p} \sum_{k=1}^{J-1} a_{tk} r_k \ .$$

In particular, for $p = n$,

$$\sum_{t=J}^{n} r'_t < r_J + \sum_{k=1}^{J-1} \sum_{t=J}^{n} a_{tk} r_k \ .$$

But now the hypothesis of the lemma, together with Lemma 2.3.3, guarantees that

$$a_{tk} \geq 0 \ , \quad k = 1,\ldots,J-1 \ , \ t = 1,\ldots,n \ .$$

Hence, our inequality becomes

$$\sum_{t=J}^{n} r'_t < r_J + \sum_{k=1}^{J-1} \sum_{t=J}^{n} a_{tk} r_k < r_J \ .$$

But (2.3.3) for m = (J - 1) is

$$\sum_{t=1}^{J-1} r'_t \leq \sum_{t=1}^{J-1} r_t \ .$$

Combining this with (2.3.2), we obtain

$$\sum_{j=1}^{n} r_j + \sum_{p=1}^{J-1} r'_p \leq \sum_{p=1}^{n} r'_p + \sum_{j=1}^{J-1} r_j \ ,$$

or

$$\sum_{j=J}^{n} r_j \leq \sum_{p=J}^{n} r'_p \ .$$

But, clearly from (2.1.2),

$$r_J \leq \sum_{j=J}^{n} r_j \ ,$$

contradicting the result,

$$\sum_{t=J}^{n} r'_t < r_J$$

derived above from the contrary supposition that no L(J) exists. Consequently, an L(J) in the range (2.3.5) must exist which satisfies (2.3.6).

For the case,

$$r_J = 0$$

Lemma 2.3.1 implies that the inequality (2.3.6) is satisfied by all p-values.

This proves that L(J) values exist and consequently we can again take $\hat{L}(J)$ to be the smallest such value. It remains to show that the structure (2.3.7) is realized for j = J.

For the case of

$$r_J = 0$$

the structure (2.3.7) is precisely that prescribed by the FIFO construction (2.2.2), so there is nothing to prove.

For $r_J > 0$, the structure (2.3.7) is identical to the FIFO construction for

$$p = 1,\ldots,J \ .$$

For p-values in the range

$$p = J+1,\ldots,\hat{L}(J)-1 \ ,$$

the FIFO construction (2.2.2) provides

$$a_{pJ} = \min\{\frac{1}{r_J}(r'_p - \sum_{k=1}^{J-1} a_{pk}r_k), 1 - \sum_{t=1}^{p-1} a_{tJ}\} \ .$$

However, by the definition of $\hat{L}(J)$, we have

$$\frac{1}{r_J}(r'_p - \sum_{k=1}^{J-1} a_{pk}r_k) < 1 - \sum_{t=1}^{p-1} a_{tJ} \ , \ p = J,\ldots,\hat{L}(J)-1 \ ,$$

so that

$$a_{pJ} = \frac{1}{r_J}(r'_p - \sum_{k=1}^{J-1} a_{pk}r_k) \ , \quad p = J+1,\ldots,\hat{L}(J)-1 \ ,$$

as required for (2.3.7). Similarly, for $p = \hat{L}(J)$, we have (2.3.6),

$$r'_{\hat{L}(J)} - \sum_{k=1}^{J-1} a_{\hat{L}(J),k} r_k \geq 1 - \sum_{t=1}^{\hat{L}(J)-1} a_{tJ} ,$$

yielding

$$a_{\hat{L}(J),J} = 1 - \sum_{t=1}^{\hat{L}(J)-1} a_{tJ} ,$$

as needed.

For $p = \hat{L}(J)+1$, we have, by what has been proved to this point,

$$0 = 1 - \sum_{t=1}^{\hat{L}(J)} a_{tJ} ,$$

and from Lemma 2.3.1,

$$r'_{\hat{L}(J)+1} - \sum_{k=1}^{J-1} a_{\hat{L}(J)+1,k} r_k \geq 0 ,$$

so that (2.2.2) yields

$$a_{\hat{L}(J)+1,k} = 0 .$$

This again retains the equality,

$$0 = 1 - \sum_{t=1}^{\hat{L}(J)+1} a_{tJ} ,$$

so that a finite number of repetitions of this argument provides

$$a_{pJ} = 0 , \quad p = \hat{L}(J)+1,\ldots,n ,$$

thereby completing the demonstration of the structure (2.3.7).

This completes the proof of the lemma.

The proof of the theorem is now at hand. Lemma 2.3.2 shows that an $\hat{L}(1)$ exists for $j = 1$ and yields the structure (2.3.7). We now make an induction on j and assume that $\hat{L}(j)$ and the structure (2.3.7) exist for

$$j = 1,\ldots,J-1 , \quad 1 < J \leq n .$$

This induction hypothesis is just the hypothesis of Lemma 2.3.4, so that the induction is completed by simply citing the conclusion of this lemma.

We then have the existence of $\hat{L}(j)$ and the structure (2.3.7) for all $j = 1,\ldots,n$. We summarize this result in the following lemma.

Lemma 2.3.5. Let r' be an n-dimensional real vector satisfying (2.3.1), (2.3.2), and (2.3.3), and let (a_{pj}) be the matrix determined by the recursion (2.2.2) applied to r and r'. Then for each j-value,

$$j = 1,\ldots,n ,$$

there exists numbers $L(j)$,

$$j \leq L(j) \leq n$$

satisfying (2.3.6). If we let $\hat{L}(j)$ denote the smallest of these numbers for each j-value, then for

$$j = 1,\ldots,n ,$$

the matrix components,

$$a_{pj} , \quad p = 1,\ldots,n ,$$

satisfy the relation (2.3.7).

Combining Lemma 2.3.3 with Lemma 2.3.5, we immediately have the conclusion that the matrix (a_{pj}) fulfills requirements (2.1.7), (2.1.8), and (2.1.9).

It only remains to show that (2.1.4) holds. From Lemma 2.3.1,

$$r'_p \geq \sum_{k=1}^{n} a_{pk} r_k , \quad p = 1,\ldots,n .$$

Suppose that one or more of these inequalities were strict. Then, on summing over p, we would have, considering the proved non-negativity of all a_{pj},

$$\sum_{p=1}^{n} r'_p > \sum_{p=1}^{n} \sum_{k=1}^{n} a_{pk} r_k = \sum_{k=1}^{n} \left(\sum_{p=1}^{n} a_{pk}\right) r_k = \sum_{k=1}^{n} r_k ,$$

where we have employed the conservation property (2.1.7) proved for (a_{pj}). However, the result

$$\sum_{p=1}^{n} r'_p > \sum_{k=1}^{n} r_k$$

contradicts the hypothesized condition (2.3.2) of the theorem. Hence, the supposition that one or more of these inequalities are strict must be false, and

$$r_p = \sum_{k=1}^{n} a_{pk} r_k \ , \quad p = 1,\ldots,n$$

which is just (2.1.4).

Thus, we have shown that the FIFO construction (2.2.2) applied to r and r' determines a matrix (a_{pj}) meeting the requirements for membership in $\mathcal{A}$. Moreover, this deferral matrix yields r' via the relation (2.1.4). By Definition 2.1.2, this demonstrates that an n-dimensional real vector r' satisfying (2.3.1), (2.3.2), and (2.3.3) must belong to $\mathcal{R}$.

This shows that (2.3.1), (2.3.2), and (2.3.3) are sufficient conditions for membership in $\mathcal{R}$, and the proof of the theorem is complete.

Theorem 2.3.2 (The FIFO Construction Determines a FIFO Policy). Let r be a demand schedule and let r' be an element in the associated set $\mathcal{R}$. Let (a_{pj}) be the deferral matrix determined by applying the FIFO construction (2.2.2) to r and r'. Then (a_{pj}) satisfies (2.2.1).

Proof: Let b and c be any two integers,

$$1 \leq b < c \leq n \ ,$$

for which $r_b > 0$, $r_c > 0$, and

$$r_j = 0 , \quad j = b+1,\ldots,c-1 .$$

Define

$$(2.3.9) \quad L_{min}(j) = \min \{p: a_{pj} > 0\} , \quad j = 1,\ldots,n ,$$

and

$$(2.3.10) \quad L_{max}(j) = \max \{p: a_{pj} > 0\} , \quad j = 1,\ldots,n .$$

It is clear that the theorem will be proved if we can demonstrate the result,

$$L_{max}(b) \leq L_{min}(c) .$$

From Lemma 2.3.5, we have that (a_{pj}) satisfies (2.3.7). Consequently, since $r_b > 0$,

$$a_{pb} = \frac{1}{r_b}(r'_p - \sum_{k=1}^{b-1} a_{pk}r_k) , \quad p = b,\ldots,\hat{L}(b)-1 ,$$

and

$$r'_p = \sum_{k=1}^{b} a_{pk}r_k = \sum_{k=1}^{c-1} a_{pk}r_k , \quad p = b,\ldots,\hat{L}(b)-1 .$$

Using this result and (2.3.7) for $j = c$, we have

$$a_{pc} = \begin{cases} 0 & , \quad p = 1,\ldots,c-1 \\ \frac{1}{r_c}(r'_p - \sum_{k=1}^{c-1} a_{pk}r_k) & , \quad p = c,\ldots,\hat{L}(b)-1 \end{cases}$$

$$= 0 , \quad p = 1,\ldots,\max\{c-1, \hat{L}(b)-1\} ,$$

so that

$$L_{min}(c) > \max \{c-1,\ \hat{L}(b)-1\} \geq \hat{L}(b)-1 ,$$

or

$$L_{min}(c) \geq \hat{L}(b) .$$

However, from (2.3.7) and the definition (2.3.10), we have

$$\hat{L}(b) \geq L_{max}(b)$$

which provides the needed inequality,

$$L_{min}(c) \geq \hat{L}(b) \geq L_{max}(b) ,$$

and the theorem is proved.

Theorem 2.3.3 (Essential Uniqueness of FIFO Construction). Let r be a demand schedule and let r' be an element in the associated set $\mathcal{R}$. Let (a_{pj}) be the deferral matrix determined by applying the FIFO construction (2.2.2) to r and r'. Let $(\bar{a}_{pj})$ be any element in $\mathcal{A}_F$ satisfying (2.1.4) for r and r'. Then for each j-value associated with a positive r_j,

$$\bar{a}_{pj} = a_{pj} , \quad p = 1,\ldots,n .$$

Proof: We proceed by an induction in the sequence of the recursion (2.2.2).

First, it is obvious that for $r_1 > 0$,

$$a_{11} = r_1'/r_1$$

$$= \bar{a}_1 \ .$$

Next, assume that the result holds through the $(\hat{p},\hat{j})^{\underline{th}}$ stage, i.e.,

$$a_{pj} = \bar{a}_{pj} \ , \ r_j > 0 \ , \ p = 1,\ldots,n \ , \ j = 1,\ldots,\hat{j}-1 \ ,$$

and for $r_{\hat{j}} > 0$,

$$a_{p\hat{j}} = \bar{a}_{p\hat{j}} \ , \ p = 1,\ldots,\hat{p} \ ,$$

where $1 \leq \hat{p} \leq n$ for $1 \leq \hat{j} \leq n-1$ and $1 \leq \hat{p} \leq n-1$ for $\hat{j} = n$. To complete the induction, we must show that for $r_{\hat{j}} > 0$, when $1 \leq \hat{p} \leq n-1$, $1 \leq \hat{j} \leq n$,

$$a_{\hat{p}+1,\hat{j}} = \bar{a}_{\hat{p}+1,\hat{j}} \ ,$$

and when $\hat{p} = n$, $1 \leq \hat{j} \leq n-1$,

$$a_{1,\hat{j}+1} = \bar{a}_{1,\hat{j}+1} \ .$$

The case,

$$1 \leq \hat{p} \leq \hat{j}-2 \ , \ 1 \leq \hat{j} \leq n \ ,$$

is trivial since by (2.1.9),

$$a_{\hat{p}+1,\hat{j}} = 0 = \bar{a}_{\hat{p}+1,\hat{j}} \ .$$

Lemma 2.3.5 guarantees the existence of an integer

$\hat{L}(\hat{j})$ for which (a_{pj}) satisfies (2.3.7). We may therefore next consider the case,

$$\hat{j}-1 \leq \hat{p} \leq \hat{L}(\hat{j})-2 \ , \ 1 \leq \hat{j} \leq n \ .$$

From (2.3.7) and the induction hypothesis,

$$a_{\hat{p}+1,\hat{j}} = \frac{1}{r_{\hat{j}}}(r'_{\hat{p}+1} - \sum_{k=1}^{\hat{j}-1} a_{\hat{p}+1,k} r_k)$$

$$= \frac{1}{r_{\hat{j}}}(r'_{\hat{p}+1} - \sum_{k=1}^{\hat{j}-1} \bar{a}_{\hat{p}+1,k} r_k) \ .$$

If $a_{\hat{p}+1,\hat{j}} = 0$, then

$$r'_{\hat{p}+1} = \sum_{k=1}^{\hat{j}-1} \bar{a}_{\hat{p}+1,k} r_k$$

and so by (2.1.4),

$$\bar{a}_{\hat{p}+1,\hat{j}} = 0 \ .$$

On the other hand, if $a_{\hat{p}+1,\hat{j}} > 0$, then

$$r'_{\hat{p}+1} = \sum_{k=1}^{\hat{j}-1} \bar{a}_{\hat{p}+1,k} r_k + a_{\hat{p}+1,\hat{j}} r_{\hat{j}} = \sum_{k=1}^{n} \bar{a}_{\hat{p}+1,k} r_k \ ,$$

so that

$$(a_{\hat{p}+1,\hat{j}} - \bar{a}_{\hat{p}+1,\hat{j}}) = \frac{1}{r_{\hat{j}}} \sum_{k=\hat{j}+1}^{n} a_{\hat{p}+1,k} r_k \geq 0 \ .$$

Suppose

$$a_{\hat{p}+1,\hat{j}} > \bar{a}_{\hat{p}+1,\hat{j}} .$$

This would imply that for some $\hat{k}$, $\hat{j}+1 \leq \hat{k} \leq n$,

$$\bar{a}_{\hat{p}+1,\hat{k}} r_{\hat{k}} > 0 ,$$

or

$$\min\{p: \bar{a}_{p\hat{k}} > 0\} \leq \hat{p}+1 .$$

By (2.1.7) and (2.1.8),

$$1 \geq \sum_{t=1}^{\hat{p}+1} a_{t\hat{j}} = a_{\hat{p}+1,\hat{j}} + \sum_{t=1}^{\hat{p}} \bar{a}_{t\hat{j}} ,$$

and we see that the supposition,

$$a_{\hat{p}+1,\hat{j}} > \bar{a}_{\hat{p}+1,\hat{j}}$$

would also imply

$$1 > \sum_{t=1}^{\hat{p}+1} \bar{a}_{t\hat{j}} ,$$

or

$$\max\{p: \bar{a}_{p\hat{j}} > 0\} > \hat{p}+1 .$$

The supposition,

$$a_{\hat{p}+1,\hat{j}} > \bar{a}_{\hat{p}+1,\hat{j}} ,$$

thus leads to a violation of the FIFO property (2.2.1) of $(\bar{a}_{pj})$. Hence, this supposition must be false and we have

$$a_{\hat{p}+1,\hat{j}} = \bar{a}_{\hat{p}+1,\hat{j}} .$$

For the case,

$$\hat{p} = \hat{L}(\hat{j})-1 , \quad 1 \leq \hat{j} \leq n ,$$

we obtain from (2.3.7) and the induction hypothesis,

$$a_{\hat{p}+1,\hat{j}} = 1 - \sum_{t=1}^{\hat{p}} a_{t\hat{j}} = 1 - \sum_{t=1}^{\hat{p}} \bar{a}_{t\hat{j}}$$

$$= \bar{a}_{\hat{p}+1,\hat{j}} + \sum_{t=\hat{p}+2}^{n} \bar{a}_{t,\hat{j}} .$$

This in turn implies that

$$(a_{\hat{p}+1,\hat{j}} - \bar{a}_{\hat{p}+1,\hat{j}}) = \sum_{t=\hat{p}+2}^{n} \bar{a}_{t\hat{j}} \geq 0 .$$

We again suppose

$$a_{\hat{p}+1,\hat{j}} > \bar{a}_{\hat{p}+1,\hat{j}} .$$

Then clearly,

$$\max\{t: \bar{a}_{t\hat{j}} > 0\} \geq \hat{p} + 2 .$$

But from Lemma 2.3.1 and the above supposition,

$$r'_{\hat{p}+1} \geq \sum_{k=1}^{\hat{j}} a_{\hat{p}+1,k} r_k > \sum_{k=1}^{\hat{j}} \bar{a}_{\hat{p}+1,k} r_k ,$$

which implies that there must exist a $\hat{k}$, $\hat{j}+1 \leq \hat{k} \leq n$, such that $r_{\hat{k}} > 0$ and

$$\min\{p: \bar{a}_{p\hat{k}} > 0\} \leq p+1 .$$

This again contradicts the FIFO property (2.2.1) of $(\bar{a}_{pj})$. Therefore the supposition must be false, and

$$a_{\hat{p}+1,\hat{j}} = \bar{a}_{\hat{p}+1,\hat{j}} , \quad \hat{p} = \hat{L}(\hat{j})-1 , \quad 1 \leq \hat{j} \leq n .$$

For the case,

$$\hat{L}(\hat{j}) \leq \hat{p} \leq n-1 , \quad 1 \leq \hat{j} \leq n ,$$

we have from (2.3.7),

$$a_{\hat{p}+1,\hat{j}} = 0 .$$

From (2.3.7) and (2.1.7),

$$1 = \sum_{t=1}^{\hat{L}(\hat{j})} a_{t\hat{j}} = \sum_{t=1}^{\hat{L}(\hat{j})} \bar{a}_{t\hat{j}} = \sum_{t=1}^{n} \bar{a}_{t\hat{j}} ,$$

which implies

$$\bar{a}_{\hat{p}+1,\hat{j}} = 0 .$$

Finally, for the last case,

$$\hat{p} = n , \quad 1 \leq \hat{j} \leq n-1 ,$$

we immediately have from (2.1.9),

$$a_{1,\hat{j}+1} = 0 = \bar{a}_{1,\hat{j}+1} .$$

This completes the induction and the theorem is proved.

2.4. Optimization in the Set $\mathcal{R}$

For the basic model discussed in this section, there exists a simple technique for selection of the optimal element in $\mathcal{R}$. Consider the n-dimensional vector r' determined by

$$(2.4.1) \quad \hat{r}'_j = \min\{x_j + \sum_{k=1}^{j-1} (x_k - \hat{r}'_k), r_j + \sum_{k=1}^{j-1} (r_k - \hat{r}'_k)\} ,$$

$$j = 1,\ldots,n ,$$

where the recursion starts with $j = 1$ and proceeds until $j = n$.

Theorem 2.4.1 ($\hat{r}'$ is an Element in $\mathcal{R}$). Let r be a demand schedule and let $\mathcal{R}$ be the associated set of feasible deferred demand schedules. If $\hat{r}'$ is the vector determined by (2.4.1), then $\hat{r}'$ is an element in $\mathcal{R}$.

Proof: By Theorem 2.3.1 we need only show that $\hat{r}'$ satisfies conditions (2.3.1), (2.3.2), and (2.3.3).

To show (2.3.1), we proceed by induction. For $j = 1$,

$$\hat{r}_1 = \min\{x_1, r_1\} \geq 0 .$$

Assume now that for $1 \leq m < n$,

$$\hat{r}'_j \geq 0 , \quad j = 1,\ldots,m .$$

Then from (2.4.1),

$$\hat{r}'_m = \min\{x_m + \sum_{k=1}^{m-1} (x_k - \hat{r}'_k), r_m + \sum_{k=1}^{m-1} (r_k - \hat{r}'_k)\} \geq 0 ,$$

and so

$$\min\{\sum_{k=1}^{m} (x_k - \hat{r}'_k), \sum_{k=1}^{m} (r_k - r'_k)\} \geq 0 .$$

Consequently,

$$\hat{r}'_{m+1} = \min\{x_{m+1} + \sum_{k=1}^{m} (x_k - r'_k), r_{m+1} + \sum_{k=1}^{m} (r_k - r'_k)\}$$

$$\geq \min\{x_{m+1}, r_{m+1}\} \geq 0$$

and the induction is completed.

Condition (2.3.2) is clearly seen to be satisfied by using (2.1.3) in conjunction with (2.4.1) for $j = n$,

$$\hat{r}'_n = \min\{x_n + \sum_{k=1}^{n-1} (x_k - \hat{r}'_k), r_n + \sum_{k=1}^{n-1} (r_k - \hat{r}'_k)\}$$

$$= \min\{\sum_{k=1}^{n} x_k, \sum_{k=1}^{n} r_k\} - \sum_{k=1}^{n-1} \hat{r}'_k$$

$$= \sum_{k=1}^{n} r_k - \sum_{k=1}^{n-1} \hat{r}'_k .$$

Finally, (2.3.3) follows immediately since by (2.4.1),

$$\hat{r}'_j \leq r_j + \sum_{k=1}^{j-1} (r_k - \hat{r}_k) , \quad j = 1,\ldots,n ,$$

so that

$$\sum_{k=1}^{j} \hat{r}_k \leq \sum_{k=1}^{j} r_k , \quad j = 1,\ldots,n .$$

This completes the proof of the theorem.

Theorem 2.4.2 ($\hat{r}'$ is the Unique Optimal Element in $\mathcal{R}$). Let r be a demand schedule and let $\mathcal{R}$ be the associated set of feasible deferred demand schedules. Let the holding and shortage cost functions satisfy the following conditions,

$$(2.4.2) \qquad h_j(0) = 0 , \quad j = 1,\ldots,n ,$$

$$(2.4.3) \quad 0 < h_j(u_1) < h_j(u_2) , \; 0 < u_1 < u_2 < \infty , \; j = 1,\ldots,n ,$$

$$(2.4.4) \quad 0 < h_j(u) , \; -\infty < u < 0 , \; j = 1,\ldots,n .$$

Then the vector $\hat{r}'$ determined by (2.4.1) is the unique element minimizing G_0 in $\mathcal{R}$.

Proof: Let y be the net stock level associated by (2.1.5) with an element r' in $\mathcal{R}$, and let $\hat{y}$ be the net stock level for $\hat{r}'$. Define

$$y_j^{\oplus} = \max\{0, y_j\} , \quad j = 1,\ldots,n ,$$

and

$$y_j^{\ominus} = \min\{0, y_j\} , \quad j = 1,\ldots,n .$$

Then the $j^{\underline{th}}$ period holding and shortage cost incurred by a deferred demand schedule r' is

$$h_j(y_j) = h_j(y_j^{\oplus}) + h_j(y_j^{\ominus}) .$$

Similarly, for the deferred demand schedule $\hat{r}'$, we have

$$h_j(\hat{y}_j) = h_j(\hat{y}_j^{\oplus}) + h_j(\hat{y}_j^{\ominus}) .$$

However, by (2.4.1),

$$\hat{r}_j' \leq x_j + \sum_{k=1}^{j-1} (x_k - \hat{r}_k') ,$$

so that

$$\hat{y}_j = \sum_{k=1}^{j} (x_k - \hat{r}_k') \geq 0 , \quad j = 1,\ldots,n .$$

Consequently, by (2.4.2),

$$h_j(\hat{y}_j) = h_j(\hat{y}_j^{\oplus}) + h_j(\hat{y}_j^{\ominus}) = h_j(\hat{y}_j^{\oplus}) ,$$

$$j = 1,\ldots,n .$$

On the other hand, by (2.4.1) and (2.3.3),

$$\hat{y}_j^{\oplus} = \hat{y}_j$$

$$= \sum_{k=1}^{j} x_k - \sum_{k=1}^{j-1} \hat{r}'_k - \hat{r}'_j$$

$$= \sum_{k=1}^{j} x_k - \sum_{k=1}^{j-1} \hat{r}'_k - \min\{\sum_{k=1}^{j} x_k - \sum_{k=1}^{j-1} \hat{r}_k, \sum_{k=1}^{j} r_k - \sum_{k=1}^{j-1} \hat{r}_k\}$$

$$= \sum_{k=1}^{j} x_k - \min\{\sum_{k=1}^{j} x_k, \sum_{k=1}^{j} r_k\}$$

$$= \max\{0, \sum_{k=1}^{j} (x_k - r_k)\}$$

$$\leq \max\{0, \sum_{k=1}^{j} (x_k - r'_k)\} = y_j^{\oplus} , \qquad j = 1,\ldots,n .$$

By (2.4.3), we must then have

$$h_j(\hat{y}_j) = h_j(\hat{y}_j^{\oplus}) \leq h_j(y_j^{\oplus}) \leq h_j(y_j) , \quad j = 1,\ldots,n ,$$

and so clearly

$$\sum_{j=1}^{n} h_j(\hat{y}_j) \leq \sum_{j=1}^{n} h_j(y_j) .$$

Thus, the deferred demand schedule r' is optimal in $\mathcal{R}$.

Now assume r' is another optimal element in $\mathcal{R}$. Then from the above result and (2.4.3), we must have

$$h_j(\hat{y}_j) = h_j(y_j) , \quad j = 1,\ldots,n ,$$

which implies

$$\hat{y}_j^{\oplus} = \hat{y}_j^{\oplus} , \quad j = 1,\ldots,n ,$$

and

$$h_j(y_j^{\ominus}) = 0 , \quad j = 1,\ldots,n ,$$

which by (2.4.4) implies

$$y_j^{\ominus} = 0 , \quad j = 1,\ldots,n .$$

Hence,

$$\hat{y}_j = \hat{y}_j^{\oplus} = y_j^{\oplus} = y_j , \quad j = 1,\ldots,n .$$

By (2.1.5), this in turn implies

$$\hat{r}'_j = r'_j , \quad j = 1,\ldots,n ,$$

and any optimal r' is thus seen to be identical with $\hat{r}'$. This completes the proof of the theorem.

3. URGENCY STRUCTURE DEFERRAL PROBLEMS

3.1. Basic Model

In the preceding section, we allowed unlimited deferral without constraints. In this section, we address the far more realistic problem of deferral limited by the urgency structure of the demand.

Consider the demand element r_j. For some of the orders comprising r_j, delivery time will be far more critical than for other orders. With this in mind, we introduce the following definition:

Definition 3.1.1. An $n \times n$ real matrix (d_{uj}) is said to be an "urgency structure" if it satisfies the three conditions,

$$(3.1.1) \qquad d_{uj} \geq 0 \ , \quad u = 1,\ldots,n \ , \ j = 1,\ldots,n \ ,$$

$$(3.1.2) \qquad d_{uj} = 0 \ , \ u < j \ , \quad u = 1,\ldots,n \ , \ j = 1,\ldots,n \ ,$$

$$(3.1.3) \qquad \sum_{u=1}^{n} d_{uj} = 1 \ , \quad j = 1,\ldots,n \ .$$

We interpret d_{uj} as the fraction of the demand r_j which can wait for shipment until precisely the u^{th} period but no longer. The significance of the constraints (3.1.1), (3.1.2), and (3.1.3) is obvious.

The assignment of promise date now consists of selecting the fraction σ_{puj} of demand $d_{uj}r_j$ which is to

be given a promise date of the $p^{\underline{th}}$ period. We assume that this assignment is compatible with the urgency structure, i.e., we require that the promise date p lie in the range,

$$(3.1.4) \qquad j \leq p \leq u \ .$$

Motivated by this requirement, we quantify the concept of a promise date policy as follows:

Definition 3.1.2. An $n \times n \times n$ real array (σ_{puj}) is said to be a "feasible promise date policy" if it satisfies the four conditions,

$$(3.1.5) \qquad \sigma_{puj} \geq 0 \ , \quad p = 1,\ldots,n \ , \ u = 1,\ldots,n \ , \quad j = 1,\ldots,n \ ,$$

$$(3.1.6) \qquad \sum_{p=1}^{n} \sigma_{puj} = 1 \ , \quad u = j,\ldots,n \ , \ j = 1,\ldots,n \ ,$$

$$(3.1.7) \qquad \sigma_{puj} = 0 \ , \ p > u \ , \ p = 1,\ldots,n \ , \ u = 1,\ldots,n \ , \quad j = 1,\ldots,n \ ,$$

$$(3.1.8) \qquad \sigma_{puj} = 0 \ , \ p < j \ , \ p = 1,\ldots,n \ , \ u = 1,\ldots,n \ , \quad j = 1,\ldots,n \ .$$

The significance of these conditions is obvious. Moreover, (3.1.7) and (3.1.8) clearly imply

$$(3.1.9) \qquad \sigma_{puj} > 0 \iff j \leq p \leq u \ ,$$

which is consistent with (3.1.4).

The set of all feasible promise date policies will be denoted by Ω.

The supplier's sole decision is to select a promise date policy (σ_{puj}) from the set Ω.

The cost elements are the same as for the model of Section 2.1. However, we must relate these costs to the promise date policy (σ_{puj}). We recall that the holding/shortage cost functions h_j depend solely on the net stock level y_j,

$$y_j = \sum_{k=1}^{j} x_k - \sum_{k=1}^{j} r'_k \ , \quad j = 1,\ldots,n \ .$$

The vector r' must now be understood to be the deferred demand schedule generated by the promise date policy (σ_{puj}),

$$(3.1.10) \qquad r'_k = \sum_{j=1}^{n} \sum_{u=1}^{n} \sigma_{kuj} d_{uj} r_j \ , \ k = 1,\ldots,n \ .$$

The total cost is then a function G,

$$G: \Omega \longrightarrow \mathbb{R} \ ,$$

where

$$(3.1.11) \qquad G((\sigma_{puj})) = \sum_{j=1}^{n} c_j(x_j) + \sum_{j=1}^{n} h_j(y_j) \ .$$

An optimal policy is one which minimizes G.

The following sections are addressed to this

optimization problem. Ideally, we would like to effect a reduction similar to that achieved in Section 2.3. Towards this end, we define two promise date policies to be equivalent if they generate the same deferred demand schedule. Each equivalence class in Ω then corresponds to a single deferred demand schedule. This suggests the following definition.

Definition 3.1.3. An n-dimensional real vector r' is said to be a "feasible deferred demand schedule" (associated with a given demand schedule r and a given urgency structure (d_{uj})) if there exists an element (σ_{puj}) in Ω satisfying (3.1.10).

We denote by $\mathcal{R}((d_{uj}))$ the set of all feasible deferred demand schedules associated with a given demand schedule r and a given urgency structure (d_{uj}). Although $\mathcal{R}((d_{uj}))$ depends on r, we do not explicitly display this dependence in the notation.

Since the function G is clearly constant on each equivalence class in Ω, we may regard G as a function on the set $\mathcal{R}((d_{uj}))$. The problem thus reduces to finding an optimal vector r'^* from the set $\mathcal{R}((d_{uj}))$, and then constructing any element (σ^*_{puj}) in the equivalence class in Ω corresponding to r'^*.

In Section 3.2, we shall show how a somewhat modified version of our earlier concept of a deferral

matrix can play a central role in this overall plan. In Section 3.3, we shall develop a characterization for the subset of $\mathcal{R}((d_{uj}))$ obtained for the important special case where only FIFO promise date policies are permitted. However, unlike the case explored in Section 2, we shall see that this restriction to FIFO policies is a substantial limitation and may rule out lesser cost policies. In Section 3.4, the SCOUR construction technique is developed which is then used to obtain a characterization of the full $\mathcal{R}((d_{uj}))$ set.

3.2. The Deferral Matrix of a Promise Date Policy

The relation (3.1.10) may be written as

$$r'_k = \sum_{j=1}^{n} a_{kj} r_j \ , \quad k = 1,\ldots,n \ , \tag{3.2.1}$$

where

$$a_{kj} = \sum_{u=1}^{n} \sigma_{kuj} d_{uj} \ , \tag{3.2.2}$$

$$k = 1,\ldots,n \ , \ j = 1,\ldots,n \ .$$

This suggests the following definition.

Definition 3.2.1. An $n \times n$ real matrix (a_{kj}) which satisfies (3.2.2) is said to be the "deferral matrix" associated with the feasible promise date policy (σ_{puj}) (under the urgency structure (d_{uj})).

The set of all deferral matrices associated with promise date policies in Ω will be denoted by $\mathcal{A}((d_{uj}))$.

We observe that set $\mathcal{R}((d_{uj}))$ consists of all n-dimensional vectors satisfying the relation (3.2.1) with respect to some matrix (a_{kj}) in $\mathcal{A}((d_{uj}))$. This observation will be used in later sections to develop a characterization of $\mathcal{R}((d_{uj}))$. First, we must provide a characterization for the set $\mathcal{A}((d_{uj}))$.

Theorem 3.2.1 (Characterization of the Set $\mathcal{A}((du_j))$). Let (d_{uj}) be an urgency structure. Then the set $\mathcal{A}((d_{uj}))$ consists of all points (a_{kj}) in $\mathcal{A}$ which satisfy

$$(3.2.3)\quad \sum_{k=1}^{m} d_{kj} \leq \sum_{k=1}^{m} a_{kj} \ , \ m = 1,\ldots,n \ , \ j = 1,\ldots,n \ .$$

Proof: Let $(a_{kj}) \in \mathcal{A}((d_{uj}))$. Then by definition, there must exist a promise date policy $(\sigma_{puj}) \in \Omega$ such that (3.2.2) holds. It immediately follows that (a_{kj}) satisfies (2.1.8) and (2.1.9). Also,

$$\sum_{k=1}^{n} a_{kj} = \sum_{k=1}^{n} \sum_{u=1}^{n} \sigma_{kuj} d_{uj} = \sum_{u=j}^{n} \left(\sum_{k=1}^{n} \sigma_{kuj}\right) d_{uj}$$

$$= \sum_{u=j}^{n} d_{uj} = 1 \ , \quad j = 1,\ldots,n \ ,$$

so that (2.1.7) is satisfied. Therefore, by Definition 2.1.1, $(a_{kj}) \in \mathcal{A}$. To show that (a_{kj}) satisfies (3.2.3), we have

$$\sum_{k=1}^{m} a_{kj} = \sum_{k=1}^{m} \sum_{u=1}^{n} \sigma_{kuj} d_{uj}$$

$$= \sum_{u=1}^{m} d_{uj} \left(\sum_{k=1}^{m} \sigma_{kuj}\right) + \sum_{u=m+1}^{n} d_{uj} \left(\sum_{k=1}^{m} \sigma_{kuj}\right)$$

$$\geq \sum_{u=1}^{m} d_{uj} \left(\sum_{k=1}^{m} \sigma_{kuj}\right)$$

$$= \sum_{u=j}^{m} d_{uj} \left(\sum_{k=1}^{m} \sigma_{kuj}\right)$$

$$= \sum_{u=j}^{m} d_{uj}$$

$$= \sum_{u=1}^{m} d_{uj} , \quad m = 1,\ldots,n , \; j = 1,\ldots,n ,$$

where the next-to-last inequality was obtained from

$$\sum_{k=1}^{m} \sigma_{kuj} = 1 , \quad u = j,\ldots,m , \; j = 1,\ldots,n ,$$

using (3.1.7) and (3.1.6).

Thus, every point in $\mathcal{A}((d_{uj}))$ lies in $\mathcal{A}$ and satisfies (3.2.3). It remains to show that all points in $\mathcal{A}$ which satisfy (3.2.3) lie in $\mathcal{A}((d_{uj}))$.

Let $(a_{kj}) \in \mathcal{A}$ and satisfy (3.2.3). We must show that there exists a promise date policy $(\sigma_{puj}) \in \Omega$ such that (3.2.2) holds. We shall demonstrate the existence of (σ_{puj}) by construction. Then by Definition 3.2.1, we will have shown that $(a_{kj}) \in \mathcal{A}((d_{uj}))$.

The construction, which will be referred to as

the "Completion of the FIFO Kernels," proceeds as follows. For each fixed j-value, $1 \leq j \leq n$, we define the two (n-j+1)-dimensional vectors,

$$(3.2.4) \qquad \delta_j = (d_{nj}, d_{n-1,j}, \ldots, d_{j+1,j}, d_{jj}) ,$$

and

$$(3.2.5) \qquad \alpha_j = (a_{nj}, a_{n-1,j}, \ldots, a_{j+1,j}, a_{jj}) .$$

We then apply the FIFO construction (2.2.2) to δ_j (in the role of the demand vector) and α_j (in the role of the deferred demand schedule) to obtain the $(n-j+1) \times (n-j+1)$ matrix (λ_{puj}). The vector δ_j clearly satisfies the condition (2.1.2) to be an (n-j+1)-period demand schedule. Similarly, the vector α_j obviously satisfies the conditions (2.3.1) and (2.3.2) of a feasible deferred demand schedule relative to demand schedule δ_j. Moreover, from (3.2.3),

$$\begin{aligned} \sum_{t=1}^{m} (\alpha_j)_t &= \sum_{k=n+1-m}^{n} a_{kj} = 1 - \sum_{k=1}^{n-m} a_{kj} \\ &\leq 1 - \sum_{k=1}^{n-m} d_{kj} = \sum_{k=n+1-m}^{n} d_{kj} \\ &= \sum_{t=1}^{m} (\delta_j)_t , \quad m = 1, \ldots, n-j+1 , \end{aligned}$$

so that α_j and δ_j also satisfy (2.3.3). Thus, α_j is a feasible deferred demand schedule relative to the demand schedule δ_j. It follows that the matrix (λ_{puj}) is a

deferral matrix corresponding to α_j and δ_j. Consequently, on the basis of Theorem 2.3.1, (λ_{puj}) must satisfy (2.1.4), (2.1.7), (2.1.8), and (2.1.9), and thus the following results are obtained:

$$(3.2.6) \quad (\alpha_j)_p = \sum_{u=1}^{n-j+1} \lambda_{puj}(\delta_j)_u \ , \ p = 1,\ldots,n-j+1 \ ,$$

$$(3.2.7) \quad \sum_{p=1}^{n-j+1} \lambda_{puj} = 1 \ , \ u = 1,\ldots,n-j+1 \ ,$$

$$(3.2.8) \quad \lambda_{puj} \geq 0 \ , \ p = 1,\ldots,n-j+1 \ , \ u = 1,\ldots,n-j+1 \ ,$$

$$(3.2.9) \quad \lambda_{puj} = 0, \ p < u \ , \ p = 1,\ldots,n-j+1 \ , \ u = 1,\ldots,n-j+1 \ .$$

By repeating this construction for each j,

$$j = 1,\ldots,n \ ,$$

we obtain n (λ_{puj}) matrices. We then define the $n \times n \times n$ array (σ_{puj}) by the following: for $j \leq p \leq u$, $p = 1,\ldots,n$, $u = 1,\ldots,n$, $j = 1,\ldots,n$,

$$(3.2.10a) \quad \sigma_{puj} = \lambda_{n-p+1,n-u+1,j}$$

and for $p > u$ or $p < j$, $p = 1,\ldots,n$, $u = 1,\ldots,n$, $j = 1,\ldots,n$,

$$(3.2.10b) \quad \sigma_{puj} = 0 \ .$$

We must now show that the array (σ_{puj}) satisfies

the requirements (3.1.5), (3.1.6), (3.1.7), and (3.1.8) for membership in Ω. Requirements (3.1.7) and (3.1.8) are immediate from (3.2.10b). From (3.2.8) and (3.2.10b), we see that (3.1.5) holds. From (3.2.7) and (3.2.10b),

$$\begin{aligned}\sum_{p=1}^{n} \sigma_{puj} &= \sum_{p=j}^{n} \sigma_{puj} \\ &= \sum_{p=j}^{n} \lambda_{n-p+1,n-u+1,j} \\ &= \sum_{p=1}^{n-j+1} \lambda_{p,n-u+1,j} \\ &= 1 , \quad u = j,\ldots,n , \quad j = 1,\ldots,n ,\end{aligned}$$

so that (3.1.6) is satisfied. Thus, $(\sigma_{puj}) \in \Omega$.

To complete the proof, we must show that (σ_{puj}) is related to (a_{kj}) through (3.2.2). Using (3.2.6) and (3.2.9), we have

$$\begin{aligned}a_{kj} = (\alpha_j)_{n-k+1} &= \sum_{u=1}^{n-j+1} \lambda_{n-k+1,u,j} (\delta_j)_u \\ &= \sum_{u=1}^{n-k+1} \lambda_{n-k+1,u,j} d_{n-u+1,j} \\ &= \sum_{u=1}^{n-k+1} \sigma_{k,n-u+1,j} d_{n-u+1,j} = \sum_{u=k}^{n} \sigma_{kuj} d_{uj} \\ &= \sum_{u=1}^{n} \sigma_{kuj} d_{uj} , \quad k = j,\ldots,n , \quad j = 1,\ldots,n .\end{aligned}$$

For $k < j$, (3.2.2) is clearly satisfied because of (3.2.10b)

and (2.1.9). Hence, (3.2.2) holds and the proof is complete.

We note that the proof is based on a constructive technique--the Completion of the FIFO Kernels. Consequently, if given an optimal element r'^* in $\mathcal{R}((d_{uj}))$, we can somehow determine an element (a^*_{kj}) in $\mathcal{A}((d_{uj}))$ corresponding to r'^* through (3.2.1), then the Completion of the FIFO Kernels applied to (a^*_{kj}) will determine an optimal promise date policy (σ^*_{puj}) in Ω.

3.3. Characterization of the set $\mathcal{R}_F((d_{uj}))$

It is not uncommon that the supplier restricts consideration to promise date policies which maintain a First-In, First-Out promise date sequence. This leads to the following definition.

Definition 3.3.1. A promise date policy $(\sigma_{puj}) \in \Omega$ is said to be a "FIFO policy" (for the demand schedule r and the urgency structure (d_{uj})) if and only if for each pair of integers (j,k),

$$1 \leq j < k \leq n$$

such that $r_j > 0$ and $r_k > 0$, the following condition is satisfied,

$$\max\{p: \sigma_{puj}d_{uj}>0 \text{ for at least one integer } u,\ 1\le u\le n\}$$

(3.3.1)

$$\le \min\{p: \sigma_{puk}d_{uk}>0 \text{ for at least one integer } u,\ 1\le u\le n\}\ .$$

The set of all FIFO promise date policies for a given demand schedule r and urgency structure (d_{uj}) will be denoted by Ω_F. Although Ω_F depends on r and (d_{uj}), we do not explicitly display this dependence in the notation.

Definition 3.2.2. The set $\mathcal{A}_F((d_{uj}))$ is defined to be the class of all $n \times n$ real matrices (a_{kj}) such that (3.2.2) holds for some FIFO promise date policy $(\sigma_{puj}) \in \Omega_F$.

Let $(a_{kj}) \in \mathcal{A}_F((d_{uj}))$. Clearly, $(a_{kj}) \in \mathcal{A}((d_{uj}))$. Then there must exist a FIFO promise date policy $(\sigma_{puj}) \in \Omega_F$ satisfying (3.2.2). Condition (3.3.1) may then be written as

$$\max\{p: \sum_{u=1}^{n} \sigma_{puj}d_{uj} > 0\}$$

(3.3.2)

$$\le \min\{p: \sum_{u=1}^{n} \sigma_{puk}d_{uk} > 0\}\ ,$$

or

(3.3.3) $$\max\{p: a_{pj} > 0\} \le \min\{p: a_{pk} > 0\}\ .$$

Thus, (a_{kj}) satisfies (2.2.1) and so $(a_{kj}) \in \mathcal{A}_F$. Hence,

$$\mathcal{A}_F((d_{uj})) \subset \mathcal{A}((d_{uj})) \cap \mathcal{A}_F .$$

On the other hand, let $(a_{kj}) \in \mathcal{A}((d_{uj})) \cap \mathcal{A}_F$. Then there must exist a promise date policy $(\sigma_{puj}) \in \Omega$ satisfying (3.2.2). The FIFO condition (2.2.1) for (a_{kj}) implies (3.3.2) and consequently (3.3.1) for (σ_{puj}). Hence, $(\sigma_{puj}) \in \Omega_F$ and so $(a_{kj}) \in \mathcal{A}_F((d_{uj}))$. Thus,

$$\mathcal{A}((d_{uj})) \cap \mathcal{A}_F \subset \mathcal{A}_F((d_{uj})) ,$$

and we have proved that

$$\mathcal{A}_F((d_{uj})) = \mathcal{A}((d_{uj})) \cap \mathcal{A}_F . \tag{3.3.4}$$

In other words, the set $\mathcal{A}_F((d_{uj}))$ consists of all matrices in $\mathcal{A}((d_{uj}))$ which satisfy (2.2.1).

Definition 3.3.3. An n-dimensional real vector r' is said to be a feasible FIFO deferred demand schedule (for the demand schedule r and the urgency structure (d_{uj})) if and only if (3.1.10) is satisfied for some FIFO promise date policy $(\sigma_{puj}) \in \Omega_F$.

We denote by $\mathcal{R}_F((d_{uj}))$ the set of all feasible FIFO deferred demand schedules.

Let $r' \in \mathcal{R}_F((d_{uj}))$ and choose $(\sigma_{puj}) \in \Omega_F$ such that (3.1.10) is satisfied. Then clearly there exists an (a_{kj}) in $\mathcal{A}_F((d_{uj}))$ such that (3.2.1) and (3.2.2)

hold. Similarly, every (a_{kj}) in $\mathcal{A}_F((d_{uj}))$ determines a vector $r' \in \mathcal{R}_F((d_{uj}))$ through (3.2.1). Consequently, we have the result that $\mathcal{R}_F((d_{uj}))$ consists of the vectors r' determined through (3.2.1) by all matrices (a_{kj}) in $\mathcal{A}_F((d_{uj}))$.

Theorem 3.3.1 (Characterization of the Set $\mathcal{R}_F((d_{uj}))$). Let $\mathcal{R}_F((d_{uj}))$ be the set of FIFO deferred demand schedules associated with the demand schedule r and the urgency structure (d_{uj}). Then $\mathcal{R}_F((d_{uj}))$ consists of all vectors in $\mathcal{R}$ which satisfy the following condition: for each pair of integers (m,M), $1 \leq m \leq M \leq n$, such that

$$\sum_{j=m}^{M} r_j \sum_{u=1}^{M} d_{uj} > 0 \tag{3.3.5a}$$

the following inequality is valid,

$$\sum_{k=1}^{M} r'_k \geq \sum_{j=m}^{M} r_j \sum_{u=1}^{M} d_{uj} + \sum_{j=1}^{m-1} r_j \, . \tag{3.3.5b}$$

Proof: First, we show that every element of $\mathcal{R}_F((d_{uj}))$ belongs to $\mathcal{R}$ and satisfies (3.3.5).

Let $r' \in \mathcal{R}_F(d_{uj})$. Then from the remark following Definition 3.3.3, there must exist a matrix $(a_{kj}) \in \mathcal{A}_F((d_{uj}))$ which satisfies (3.2.1). From (3.3.4) and Theorem 3.2.1,

$$(a_{kj}) \in \mathcal{A}((d_{uj})) \subset \mathcal{A} \, ,$$

and hence by Definition 2.1.2,

$$r' \in \mathcal{R}.$$

Let (m,M) be any pair of integers,

$$1 \leq m \leq M \leq n ,$$

for which (3.3.5a) holds. Then, there must exist an integer $\hat{j}$,

$$m \leq \hat{j} \leq M ,$$

such that

$$\sum_{u=1}^{M} d_{u\hat{j}} r_{\hat{j}} > 0 ,$$

and consequently,

$$r_{\hat{j}} > 0 ,$$

and

$$\sum_{u=1}^{M} d_{u\hat{j}} > 0 .$$

Again by Theorem 3.2.1, (a_{kj}) and (d_{uj}) must satisfy (3.2.3), so that in particular,

$$\sum_{k=1}^{M} a_{k\hat{j}} \geq \sum_{u=1}^{M} d_{u\hat{j}} > 0$$

and, using the notation (2.3.9) and (2.3.10),

$$L_{min}(\hat{j}) \leq M .$$

By (3.3.4), $(a_{kj}) \in \mathcal{A}_F$ and hence (2.2.1) holds. Thus, for every j, $1 \leq j \leq m$, such that $r_j > 0$, we have

$$L_{max}(j) \leq L_{min}(\hat{j}) \leq M.$$

This insures that

$$a_{kj} r_j = 0 , \quad k = M,\ldots,n ,$$

and so we may write

$$r_j = r_j \sum_{k=1}^{n} a_{kj} = r_j \sum_{k=1}^{M} a_{kj} , \quad j = 1,\ldots,m-1 ,$$

whether or not $r_j > 0$. Summing over (3.2.1) and using the preceding equality, we obtain (3.3.5b),

$$\begin{aligned}
\sum_{k=1}^{M} r'_k &\geq \sum_{k=1}^{M} \sum_{j=1}^{n} a_{kj} r_j \\
&\geq \sum_{k=1}^{M} \sum_{j=1}^{M} a_{kj} r_j \\
&= \sum_{j=1}^{m-1} \sum_{k=1}^{M} a_{kj} r_j + \sum_{j=m}^{M} \sum_{k=1}^{M} a_{kj} r_j \\
&= \sum_{j=1}^{m-1} r_j + \sum_{j=m}^{M} \sum_{k=1}^{M} a_{kj} r_j \\
&\geq \sum_{j=1}^{m-1} r_j + \sum_{j=m}^{M} \sum_{k=1}^{M} d_{kj} r_j ,
\end{aligned}$$

where the last inequality follows from (3.2.3). This completes the proof that r' lies in $\mathcal{R}$ and satisfies (3.3.5).

It now remains to demonstrate that a vector r' which belongs to $\mathcal{R}$ and satisfies (3.3.5) must lie in $\mathcal{R}_F((d_{uj}))$. By Theorem 2.3.2, we know that the FIFO

construction (2.2.2) applied to r and r' determines a deferral matrix $(a_{kj}) \in \mathcal{A}_F$ satisfying (2.1.4). We now show that (3.2.3) also holds for (a_{kj}) and hence $(a_{kj}) \in \mathcal{A}_F((d_{uj}))$ by Theorem 3.2.1 and (3.3.4).

To demonstrate (3.2.3) for (a_{kj}), it suffices to show that for any pair of integers (m,M), $1 \leq m \leq M \leq n$, the following inequality is valid,

$$\sum_{u=m}^{M} d_{um} \leq \sum_{u=m}^{M} a_{um} \quad . \tag{3.3.6}$$

For $r_m = 0$, (2.2.2a) immediately provides the needed result,

$$\sum_{u=m}^{M} a_{um} = 1 \geq \sum_{u=m}^{M} d_{um} \quad .$$

For $r_m > 0$, either (3.3.5a) must hold or

$$\sum_{j=m}^{M} r_j \sum_{u=1}^{M} d_{uj} = 0 \quad .$$

In the latter case,

$$\sum_{u=1}^{M} d_{um} = 0 \quad ,$$

and (3.3.6) is immediate. On the other hand, when (3.3.5a) holds, then (3.3.5b) must also be true and we have

$$0 \leq \sum_{k=1}^{M} r'_k - [\sum_{j=m}^{M} r_j \sum_{u=1}^{M} d_{uj} + \sum_{j=1}^{m-1} r_j]$$

$$= \sum_{k=1}^{M} \sum_{j=1}^{n} a_{kj} r_j - [\sum_{j=m}^{M} r_j \sum_{u=1}^{M} d_{uj} + \sum_{j=1}^{m-1} r_j]$$

$$= \sum_{k=1}^{M} \sum_{j=1}^{M} a_{kj} r_j - [r_m \sum_{u=1}^{M} d_{um} + \sum_{j=m+1}^{M} r_j \sum_{u=1}^{M} d_{uj} + \sum_{j=1}^{m-1} r_j]$$

$$= r_m[\sum_{k=1}^{M} a_{km} - \sum_{u=1}^{M} d_{um}] + \sum_{j=m+1}^{M} r_j[\sum_{k=1}^{M} a_{kj} - \sum_{u=1}^{M} d_{uj}]$$

$$+ \sum_{j=1}^{m-1} r_j[\sum_{k=1}^{M} a_{kj} - 1] ,$$

or

$$r_m[\sum_{k=1}^{M} a_{km} - \sum_{u=1}^{M} d_{um}]$$

$$\geq \sum_{j=m+1}^{M} r_j[\sum_{u=1}^{M} d_{uj} - \sum_{k=1}^{M} a_{kj}] + \sum_{j=1}^{m-1} r_j[1 - \sum_{k=1}^{M} a_{kj}]$$

$$\geq \sum_{j=m+1}^{M} r_j[\sum_{u=1}^{M} d_{uj} - \sum_{k=1}^{M} a_{kj}]$$

$$= \sum_{\substack{j=m+1 \\ r_j>0}}^{M} r_j[\sum_{u=1}^{M} d_{uj} - \sum_{k=1}^{M} a_{kj}] .$$

Since (a_{kj}) satisfies the FIFO condition (2.2.1) and since we have assumed $r_m > 0$, it follows that for any integer j in the range of the sum on the right hand side of the last equation, i.e., $m < j \leq M$ and $r_j > 0$,

the following must be true,

$$L_{max}(m) \leq L_{min}(j) \ .$$

We must now consider two cases. First, when

$$L_{max}(m) > M \ ,$$

we have

$$L_{min}(j) > M$$

and

$$a_{kj} = 0 \ , \quad k = 1,\ldots,M \ .$$

The inequality (3.3.7) then becomes

$$r_m[\sum_{k=1}^{M} a_{km} - \sum_{u=1}^{M} d_{um}] \geq 0 \ ,$$

and since $r_m > 0$, we have the desired result (3.3.6),

$$\sum_{k=1}^{M} a_{km} \geq \sum_{u=1}^{M} d_{um} \ .$$

For the second case,

$$L_{max}(m) \leq M \ ,$$

we have immediately,

$$\sum_{k=1}^{M} a_{km} = 1 \geq \sum_{u=1}^{M} d_{um} \ .$$

Thus, (a_{kj}) satisfies (3.3.6), and we have proved that $(a_{kj}) \in \mathcal{A}_F((d_{uj}))$.

Finally, since r' and (a_{kj}) satisfy (3.2.1), it is immediate from the remark following Definition 3.3.3 that $r' \in \mathcal{R}_F((d_{uj}))$. This completes the proof.

Example 3.3.1: A Non-FIFO Policy.

Consider the problem of demand r,

$$r_1 = 1 , \quad r_2 = 1 , \quad r_3 = 1 ,$$

with urgency structure,

$$(d_{uj}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} .$$

By (σ_{puj}) we denote the promise date policy which gives all first period demand a third period promise and promises immediate shipment to all other demand. This policy (σ_{puj}) yields the deferral matrix (a_{kj}) through (3.2.2), where

$$(a_{kj}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} .$$

Since (a_{kj}) violates the FIFO property (2.2.1),

$$\max\{k: a_{k1} > 0\} = 3 \not\leq \min\{k: a_{k2} > 0\} = 2 ,$$

we see that (a_{kj}) lies in

$$\mathcal{A}((d_{uj})) - \mathcal{A}_F((d_{uj}))$$

and, consequently, (σ_{puj}) must lie on

$$\Omega - \Omega_F .$$

The deferred demand schedule r' associated with (a_{kj}) by (3.2.1) has components,

$$r'_1 = 0 , \quad r'_2 = 1 , \quad r'_3 = 2 .$$

For m = 2, M = 2, condition (3.3.5a) is fulfilled.

$$\sum_{j=2}^{2} r_j \sum_{u=1}^{2} d_{uj} = 1 > 0 ,$$

but not condition (3.3.5b),

$$\sum_{k=1}^{2} r'_k = 1 \not\leq \sum_{j=2}^{2} r_j \sum_{u=1}^{2} d_{uj} + r_1 = 1 + 1 = 2 .$$

Hence, from Theorem 3.3.1, $r' \notin \mathcal{A}_F((d_{uj}))$.

To gain further insight into this situation, we apply the FIFO construction (2.2.2) to r and r' to obtain the deferral matrix $(\hat{a}_{kj})$,

$$(\hat{a}_{kj}) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} .$$

By inspection, we see that $(\hat{a}_{kj})$ violates the condition (3.2.3),

$$\sum_{k=1}^{2} \hat{a}_{k2} = 0 \neq \sum_{u=1}^{2} d_{u2} = 1 ,$$

so that $(\hat{a}_{kj})$ cannot belong to $\mathcal{A}((d_{uj}))$.

This example shows that the imposition of the FIFO constraint is a material restriction, i.e., if we were to limit our optimization search to $\mathcal{R}_F((d_{uj}))$, the perfectly valid (albeit non-FIFO) policy (σ_{puj}) would not be considered.

In addition, we see that the FIFO construction (2.2.2) applied to an element r' outside of $\mathcal{R}_F((d_{uj}))$ can yield an infeasible deferral matrix $(\hat{a}_{kj})$, i.e., one not belonging to $\mathcal{A}((d_{uj}))$. This indicates that the FIFO construction cannot be meaningfully applied to all elements of $\mathcal{R}((d_{uj}))$. Therefore, if we should want to drop the imposed FIFO condition and extend our optimization search to all of $\mathcal{R}((d_{uj}))$, then we will clearly need a new method for the construction of the deferral matrix.

3.4. Characterization of the Set $\mathcal{R}((d_{uj}))$.

In Section 3.1, it was shown that the optimization problem can be solved by first finding an element r'* which minimizes G over the set $\mathcal{R}((d_{uj}))$, and then determining any promise date policy (σ^*_{puj}) in the equivalence class in $\mathcal{O}$ corresponding to r'*. To

implement this approach, we require a characterization of $\mathcal{R}((d_{uj}))$ specifying that subset of $\mathcal{R}^n$ in which to conduct the search for r'^*. In addition, once r'^* is found, by some optimization technique, a constructive procedure is needed for determining an optimal promise date policy (σ^*_{puj}) associated with r'^*. The following theorem supplies both these requirements.

Theorem 3.4.1 (Characterization of the Set $\mathcal{R}((d_{uj}))$). Let r be a demand schedule and let (d_{uj}) be an urgency structure. Then the set $\mathcal{R}((d_{uj}))$ consists of all vectors r' in $\mathcal{R}$ which satisfy

$$(3.4.1) \qquad \sum_{k=m}^{M} r'_k \geq \sum_{j=m}^{M} r_j \sum_{u=1}^{M} d_{uj} \ , \ 1 \leq m \leq M \leq n \ .$$

Proof: Let $r' \in \mathcal{R}((d_{uj}))$. Then by the remark following Definition 3.2.1, there must exist a deferral matrix $(a_{kj}) \in \mathcal{A}((d_{uj}))$ satisfying (3.2.1). By Theorem 3.2.1, $(a_{kj}) \in \mathcal{A}$, and we have at once that $r' \in \mathcal{R}$. Moreover, using (3.2.3), we show that r' satisfies (3.4.1),

$$\sum_{k=m}^{M} r'_k = \sum_{k=m}^{M} \sum_{j=1}^{n} a_{kj} r_j \geq \sum_{k=m}^{M} \sum_{j=m}^{M} a_{kj} r_j$$

$$= \sum_{j=m}^{M} r_j \sum_{k=m}^{M} a_{kj} = \sum_{j=m}^{M} r_j \sum_{k=1}^{M} a_{kj}$$

$$\geq \sum_{j=m}^{M} r_j \sum_{k=1}^{M} d_{kj} \ , \ 1 \leq m \leq M \leq n \ .$$

It remains to show that every vector r' in $\mathcal{R}$ which satisfies (3.4.1) must lie in $\mathcal{R}((d_{uj}))$. The proof of this result is based upon a construction which develops a deferral matrix (a_{kj}) belonging to $\mathcal{A}((d_{uj}))$ and satisfying (3.2.1). This immediately leads to the desired result, $r' \in \mathcal{R}((d_{uj}))$. Moreover, this construction followed by a Completion of FIFO Kernels on (a_{kj}) provides the constructive procedure needed to determine a promise date policy (σ_{puj}) associated with the deferred demand schedule r'.

We shall refer to the construction for determining (a_{kj}) as "SCOUR" (Scheduling for Control of Outstanding Urgency Requirements). Before proceeding to a formal statement of the SCOUR recursion, we present the following heuristic argument.

Suppose we have already determined

$$a_{tj} , \quad t = 1,\ldots,k-1 , \quad j = 1,\ldots,n .$$

We now wish to compute

$$a_{kj} , \quad j = 1,\ldots,n .$$

From (2.1.9), the components for $j > k$ must be zero, so we need only be concerned with $j \leq k$. We shall say that our choice of a_{kj} fulfills the $L^{\underline{th}}$ cycle urgency requirement of the $j^{\underline{th}}$ period demand if and only if

$$\sum_{u=1}^{k+L-1} d_{uj} \leq \sum_{t=1}^{k} a_{tj} \ .$$

Clearly, to insure this, a $k^{\underline{th}}$ period shipment promise for $j^{\underline{th}}$ period demand must be made amounting to at least

$$\left[\sum_{u=1}^{k+L-1} d_{uj} - \sum_{t=1}^{k-1} a_{tj} \right]^{+} r_j \ ,$$

where "$[\ ]^{+}$" indicates the positive part of the quantity within the brackets. SCOUR maximizes the smallest L fulfilled for all j, $1 \leq j \leq k$. If this value is $(L_k - 1)$, then this would entail promises for $k^{\underline{th}}$ period shipment in the amount of

$$\sum_{s=1}^{k} \left[\sum_{u=1}^{k+L_k-2} d_{us} - \sum_{t=1}^{k-1} a_{ts} \right]^{+} r_s \ .$$

Suppose additional $k^{\underline{th}}$ period promises are possible under the deferred demand schedule, i.e.,

$$r_k' > \sum_{s=1}^{k} \left[\sum_{u=1}^{k+L_k-2} d_{us} - \sum_{t=1}^{k-1} a_{ts} \right]^{+} r_s \ .$$

SCOUR assigns these additional promises so as to fulfill the $L_k^{\underline{th}}$ cycle urgency requirement first for the $1^{\underline{st}}$ period demand, then for the $2^{\underline{nd}}$ period demand, etc., until a Terminating Condition is reached for the $J_k^{\underline{th}}$ period demand,

$$r'_k \le \sum_{s=1}^{J_k} \Big[\sum_{u=1}^{k+L_k-1} d_{us} - \sum_{t=1}^{k-1} a_{ts} \Big]^+ r_s$$

$$+ \sum_{s=J_k+1}^{k} \Big[\sum_{u=1}^{k+L_k-2} d_{us} - \sum_{t=1}^{k-1} a_{ts} \Big]^+ r_s \ .$$

The component (a_{kJ_k}) is then selected so that the total demand promised for $k^{\underline{th}}$ period shipment equals r'_k.

We now give the formal statement of the SCOUR construction. Suppose that the quantities,

$$a_{tj} \ , \quad t = 1,\ldots,k-1 \ , \ j = 1,\ldots,n \ ,$$

have been previously determined. We assume the integer pairs (L,J),

$$(3.4.2) \qquad 1 \le L \le n-k+1 \ , \quad 1 \le J \le k \ ,$$

to be ordered lexicographically. We define the pair (L_k,J_k) to be the smallest pair in this set for which the following Terminating Condition is realized,

$$(3.4.3) \qquad r'_k \le \sum_{s=1}^{J_k} \Big[\sum_{u=1}^{k+L_k-1} d_{us} - \sum_{t=1}^{k-1} a_{ts} \Big]^+ r_s$$

$$+ \sum_{s=J_k+1}^{k} \Big[\sum_{u=1}^{k+L_k-2} d_{us} - \sum_{t=1}^{k-1} a_{ts} \Big]^+ r_s \ .$$

Assuming that (3.4.3) is satisfied for some (L_k,J_k), the quantities a_{kj}, $1 \le j \le n$, can now be determined as follows:

for $r_j = 0$,

$$(3.4.4a) \qquad a_{kj} = \begin{cases} 0 \ , & j = 1,\ldots,k-1 \\ 1 \ , & j = k \\ 0 \ , & j = k+1,\ldots,n \ , \end{cases}$$

and for $r_j > 0$,

$$(3.4.4b) \quad a_{kj} = \begin{cases} [\sum_{u=1}^{k+L_k-1} d_{uj} - \sum_{t=1}^{k-1} a_{tj}]^+ \ , & j = 1,\ldots,J_k-1 \\ \frac{1}{r_{J_k}}(r'_k - \sum_{s=1}^{J_k-1} [\sum_{u=1}^{k+L_k-1} d_{uj} - \sum_{t=1}^{k-1} a_{tj}]^+ r_s \\ \quad - \sum_{s=J_k+1}^{k} [\sum_{u=1}^{k+L_k-2} d_{uj} - \sum_{t=1}^{k-1} a_{tj}]^+ r_s), & j = J_k \\ [\sum_{u=1}^{k+L_k-2} d_{uj} - \sum_{t=1}^{k-1} a_{tj}]^+ \ , & j = J_k+1,\ldots,k \\ 0 \ , & j = k+1,\ldots,n \ . \end{cases}$$

We observe that (3.4.4) determines quantities a_{1j}, $1 \leq j \leq n$, in terms of r, (d_{uj}), and r'. The computation (3.4.4) can then be applied recursively for $k = 2,3,\ldots,n$, to obtain the SCOUR matrix (a_{kj}).

For $r' \in \mathcal{R}$ and r' satisfying (3.4.1), we must show first, the Terminating Condition (3.4.3) is fulfilled for each k, $1 \leq k \leq n$, so that (3.4.4) does in fact

determine a matrix (a_{kj}), and secondly,

$$(a_{kj}) \in \mathcal{A}((d_{uj})) .$$

Lemma 3.4.1. Let $r' \in \mathcal{R}$ and satisfy (3.4.1). Then (1) there exists a smallest integer pair (L_1, J_1),

$$1 \leq L_1 \leq n , \quad J_1 = 1 ,$$

for which (3.4.3) is satisfied, and (2) the quantities a_{1j}, $1 \leq j \leq n$, determined from (3.4.4) have the following properties:

$$a_{1j} \geq 0 \quad , \quad j = 1,\ldots,n ,$$

$$a_{1j} = 0 \quad , \quad j = 2,\ldots,n ,$$

$$a_{1j} \leq 1 \quad , \quad j = 1,\ldots,n ,$$

$$d_{1j} \leq a_{1j} , \quad j = 1,\ldots,n ,$$

$$r_1' = \sum_{j=1}^{n} a_{1j} r_j .$$

Proof: For $(L_1, J_1) = (n,1)$, the right-hand side of (3.4.3) becomes

$$\sum_{s=1}^{1} \Big[\sum_{u=1}^{n} d_{us} - \sum_{t=1}^{0} a_{ts}\Big]^{+} r_s = \Big[\sum_{u=1}^{n} d_{u1}\Big] r_1$$

$$= r_1$$

$$\geq r_1' .$$

Consequently, a smallest (L_1,J_1) must exist. For $r_1 = 0$, the conclusions of the lemma are obvious. For $r_1 > 0$, (2.4.4b) becomes

$$a_{1j} = \begin{cases} \dfrac{r_1'}{r_1} , & j = 1 \\ 0 , & j = 2,\ldots,n . \end{cases}$$

From (3.4.1) with $m = M = 1$,

$$r_1' \geq r_1 d_{11}$$

or

$$a_{11} = \frac{r_1'}{r_1} \geq d_{11} .$$

The remaining conclusions of the lemma are immediate.

Lemma 3.4.2. Let $r' \in \mathcal{R}$ and satisfy (3.4.1). Let (3.4.3) be satisfied for each k by the smallest pairs (L_k,J_k), $1 \leq k \leq I \leq n$, and let the resulting SCOUR components

$$a_{kj} , \quad k = 1,\ldots,I , \quad j = 1,\ldots,n ,$$

satisfy the conditions,

$$a_{kj} \geq 0 \ , \ k = 1,\ldots,I \ , \ j = 1,\ldots, n \ , \tag{3.4.5}$$

$$a_{kj} = 0 \ , \ k < j \ , \ k = 1,\ldots,I \ , \ j = 1,\ldots,n \ , \tag{3.4.6}$$

$$\sum_{t=1}^{m} a_{tj} \leq 1 \ , \ m = 1,\ldots,I \ , \ j = 1,\ldots,n \ , \tag{3.4.7}$$

$$\sum_{u=1}^{m} d_{uj} \leq \sum_{t=1}^{m} a_{tj} \ , \ m = 1,\ldots,I \ , \ j = 1,\ldots,n \ , \tag{3.4.8}$$

$$r'_k = \sum_{j=1}^{n} a_{kj} r_j \ , \ k = 1,\ldots,I \ . \tag{3.4.9}$$

Then there exists a smallest pair (L_{I+1}, J_{I+1}) satisfying (3.4.3) for $k = I + 1$, and the resulting SCOUR components $a_{I+1,j}$, $1 \leq j \leq n$, have the following properties:

$$a_{I+1,j} \geq 0 \ , \ j = 1,\ldots,n \ , \tag{3.4.10}$$

$$a_{I+1,j} = 0 \ , \ j = I+2,\ldots,n \ , \tag{3.4.11}$$

$$\sum_{t=1}^{m} a_{tj} \leq 1 \ , \ m = 1,\ldots,I+1 \ , \ j = 1,\ldots,n \ , \tag{3.4.12}$$

$$r'_{I+1} = \sum_{j=1}^{n} a_{I+1,j} r_j \ . \tag{3.4.13}$$

Proof: The largest pair in the set (3.4.2) for $k = I + 1$ is $(n-I, I+1)$. For this pair, the right-hand side of (3.4.3) becomes

$$\sum_{s=1}^{I+1} \Big[\sum_{u=1}^{n} d_{us} - \sum_{t=1}^{I} a_{ts}\Big]^{+} r_s = \sum_{s=1}^{I+1} \Big[1 - \sum_{t=1}^{I} a_{ts}\Big] r_s$$

$$= \sum_{s=1}^{I+1} r_s - \sum_{t=1}^{I} \sum_{s=1}^{I+1} a_{ts} r_s$$

$$= \sum_{s=1}^{I+1} r_s - \sum_{t=1}^{I} r'_t$$

$$\geq r'_{I+1} .$$

Hence, (3.4.3) is satisfied by one pair in the admissible set (3.4.2) and, consequently, there must exist a smallest such pair (L_{I+1}, J_{I+1}).

The conditions,

$$a_{I+1,j} \geq 0 , \quad j = 1,\ldots,n ,$$

and

$$a_{I+1,j} = 0 , \quad j = I+2,\ldots,n ,$$

are obvious.

To obtain

$$\sum_{t=1}^{m} a_{tj} \leq 1 , \quad m = 1,\ldots,I+1 , \; j = 1,\ldots,n ,$$

it suffices to prove that

$$a_{I+1,j} \leq 1 - \sum_{t=1}^{I} a_{tj} , \quad j = 1,\ldots,n .$$

This result is obvious for $r_j = 0$. For $r_j > 0$, $j \neq J_{I+1}$, the result is immediate from (3.4.4b). For $j = J_{I+1}$, from (3.4.3) and (3.4.4b),

$$a_{I+1,J_{I+1}} \leq \Big[\sum_{u=1}^{I+L_{I+1}} d_{uJ_{I+1}} - \sum_{t=1}^{I} a_{tJ_{I+1}} \Big]^+$$

$$\leq \Big[1 - \sum_{t=1}^{I} a_{tJ_{I+1}} \Big] .$$

Finally, from (3.4.4b), we have

$$\sum_{j=1}^{n} a_{I+1,j} r_j = \sum_{j=1}^{I+1} a_{I+1,j} r_j$$

$$= a_{I+1,J_{I+1}} r_{J_{I+1}}$$

$$+ \sum_{s=1}^{J_{I+1}-1} \Big[\sum_{u=1}^{I+L_{I+1}} d_{us} - \sum_{t=1}^{I} a_{ts} \Big]^+ r_s$$

$$+ \sum_{s=J_{I+1}+1}^{I+1} \Big[\sum_{u=1}^{I+L_{I+1}-1} d_{us} - \sum_{t=1}^{I} a_{ts} \Big]^+ r_s .$$

For $r_{J_{I+1}} > 0$, (3.4.4b) provides

$$\sum_{j=1}^{n} a_{I+1,j} r_j = r'_{I+1} .$$

On the other hand, by definition of (L_{I+1}, J_{I+1}) as the smallest pair satisfying (3.4.3), the case $r_{J_{I+1}} = 0$ can

only arise when

$$L_{I+1} = 1 , \quad J_{I+1} = 1 .$$

This implies

$$r'_{I+1} \leq \sum_{s=2}^{I+1} [\sum_{u=1}^{I} d_{us} - \sum_{t=1}^{I} a_{ts}]^{+} r_s = 0$$

since the hypothesis of the lemma states that

$$\sum_{u=1}^{I} d_{uj} \leq \sum_{t=1}^{I} a_{tj} , \quad j = 1,\ldots,n .$$

We observe that this also implies $r'_{I+1} = 0$. Then for any j such that $r_j > 0$, $2 \leq j \leq I+1$,

$$a_{I+1,j} = [\sum_{u=1}^{I} d_{uj} - \sum_{t=1}^{I} a_{tj}]^{+} = 0 ,$$

so that

$$r'_{I+1} = 0 = \sum_{j=1}^{n} a_{I+1,j} r_j .$$

This completes the proof of the lemma.

Lemma 3.4.3. Let $r' \in \mathcal{R}$ and satisfy (3.4.1). Let (L_k, J_k) be the smallest pair satisfying (3.4.3) for each k, $1 \leq k \leq I < n$. Suppose that the resulting SCOUR components

$$a_{kj} , \quad k = 1,\ldots,I , \quad j = 1,\ldots,n ,$$

satisfy (3.4.5), (3.4.6), (3.4.7), (3.4.8), and (3.4.9). Suppose further that there exists j-values, $1 \leq j \leq I$, such that

$$(3.4.14) \qquad [\sum_{u=1}^{I+1} d_{uj} - \sum_{t=1}^{I} a_{tj}] r_j < 0 \ .$$

Let J(I) denote the largest such j-value. Then there exists at least one integer K(I),

$$(3.4.15) \qquad J(I) \leq K(I) \leq I \ ,$$

such that

$$(3.4.16) \quad [\sum_{u=1}^{I+1} d_{uj} - \sum_{t=1}^{I} a_{tj}] r_j = \begin{cases} \leq 0 \ , & j=1,\ldots,K(I) \\ \geq 0 \ , & J=K(I)+1,\ldots,I \ , \end{cases}$$

and

$$(3.4.17) \quad [\sum_{u=1}^{I+1} d_{uj} - \sum_{t=1}^{K(I)} a_{tj}] r_j \leq 0 \ , \ j = 1,\ldots,K(I) \ .$$

Moreover, if $K_{max}(I)$ denotes the largest of the integers satisfying (3.4.15), (3.4.16), and (3.4.17), then

$$(3.4.18) \qquad a_{kj} = 0 \ , \ k = K_{max}(I)+1,\ldots,I \ , \quad j = 1,\ldots,K_{max}(I) \ .$$

Proof: We observe first that the strictness of the defining inequality (3.4.14) for J(I) guarantees that $r_{J(I)} > 0$.

Consider the set of indices,

$$\{i: a_{iJ(I)} > 0,\ J(I) \leq i \leq I\} .$$

This set cannot be null since that would imply

$$\sum_{u=I}^{I+1} d_{uJ(I)} - \sum_{t=I}^{I} a_{tJ(I)} = \sum_{u=I}^{I+1} d_{uJ(I)} \geq 0$$

contrary to (3.4.14). Thus, we can define an integer K(I) by

$$(3.4.19)\quad K(I) = \max\{i: a_{iJ(I)} > 0,\ J(I) \leq i \leq I\} .$$

It then remains to show that this integer K(I) has the required properties (3.4.16) and (3.4.17).

At the outset, we see that (3.4.19) implies

$$a_{kJ(I)} = 0 ,\quad k = K(I)+1,\ldots,I ,$$

and so

$$\begin{aligned} 0 &> \sum_{u=I}^{I+1} d_{uJ(I)} - \sum_{t=I}^{I} a_{tJ(I)} \\ &= \sum_{u=I}^{I+1} d_{uJ(I)} - \sum_{t=I}^{K(I)} a_{tJ(I)} , \end{aligned}$$

and

$$a_{K(I),J(I)} > \sum_{u=I}^{I+1} d_{uJ(I)} - \sum_{t=I}^{K(I)-1} a_{tJ(I)} .$$

Combining this last result with the definition (3.4.19)

of K(I), we may write

$$(3.4.20)\quad a_{K(I),J(I)} > \Big[\sum_{u=I}^{I+1} d_{uJ(I)} - \sum_{t=1}^{K(I)-1} a_{tJ(I)}\Big]^{+} .$$

Now consider the Terminating Pair $(L_{K(I)}, J_{K(I)})$ for the $K(I)^{\underline{th}}$ stage. There are three possibilities:

Case 1: $J(I) < J_{K(I)}$,

Case 2: $J(I) > J_{K(I)}$,

Case 3: $J(I) = J_{K(I)}$.

For all these cases, we claim that

$$K(I) + L_{K(I)} \geq I + 3 .$$

First, consider Case 1. Since $r_{J(I)}$ must be positive as noted above, the SCOUR formula (3.4.4) provides that

$$a_{K(I),J(I)} = \Big[\sum_{u=I}^{K(I)+L_{K(I)}-1} d_{uJ(I)} - \sum_{t=1}^{K(I)-1} a_{tJ(I)}\Big]^{+}$$

in light of (3.4.20), we see at once that we must have

$$K(I) + L_{K(I)} - 1 > I + 1 ,$$

or

$$K(I) + L_{K(I)} > I + 2 .$$

Similarly, for Case 2,

$$K(I) + L_{K(I)} - 2 > I + 1 ,$$

or

$$K(I) + L_{K(I)} > I + 3 .$$

Finally, for Case 3, we observe that the definition (3.4.3) of the Terminating Pair and the SCOUR formula

(3.4.4) combine to yield the inequality

$$\left[\sum_{u=1}^{K(I)+L_{K(I)}-1} d_{uJ(I)} - \sum_{t=1}^{K(I)-1} a_{tJ(I)}\right]^{+}$$

$$\geq \frac{1}{r_{J(I)}}\left(r'_{K(I)} - \sum_{s=J(I)+1}^{K(I)}\left[\sum_{u=1}^{K(I)+L_{K(I)}-2} d_{us} - \sum_{t=1}^{K(I)-1} a_{ts}\right]^{+} r_s\right.$$

$$\left. - \sum_{s=1}^{J(I)-1}\left[\sum_{u=1}^{K(I)+L_{K(I)}-1} d_{us} - \sum_{t=1}^{K(I)-1} a_{ts}\right]^{+} r_s\right)$$

$$= a_{K(I),J(I)} \cdot$$

Applying (3.4.40) again,

$$\left[\sum_{u=1}^{K(I)+L_{K(I)}-1} d_{uJ(I)} - \sum_{t=1}^{K(I)-1} a_{tJ(I)}\right]^{+}$$

$$> \left[\sum_{u=1}^{I+1} d_{uJ(I)} - \sum_{t=1}^{K(I)-1} a_{tJ(I)}\right]^{+} ,$$

which implies

$$K(I) + L_{K(I)} - 1 > I + 1 ,$$

or

$$K(I) + L_{K(I)} > I + 2 .$$

Thus, we have proved our claim that

$$K(I) + L_{K(I)} > I + 2 ,$$

or equivalently,

$$K(I) + L_{K(I)} \geq I + 3 .$$

We now demonstrate that the following inequality holds for each k, $1 \leq k \leq I + 1$,

$$(3.4.21) \qquad a_{kj} r_j \geq \Big[\sum_{u=1}^{k+L_k-2} d_{uj} - \sum_{t=1}^{k-1} a_{tj} \Big]^+ r_j , \quad j = 1, \ldots, k ,$$

provided that

$$(L_k, J_k) > (1,1) .$$

This result is immediate from (3.4.4) for all j-values except $j = J_k$. For $j = J_k$, suppose (3.4.21) fails to hold. Then,

$$\Big[\sum_{u=1}^{k+L_k-2} d_{uJ_k} - \sum_{t=1}^{k-1} a_{tJ_k} \Big]^+ r_{J_k}$$

$$> a_{kJ_k} r_{J_k}$$

$$= r'_k - \sum_{s=J_k+1}^{k} \Big[\sum_{u=1}^{k+L_k-2} d_{us} - \sum_{t=1}^{k-1} a_{ts} \Big]^+ r_s$$

$$- \sum_{s=1}^{J_{k-1}} \Big[\sum_{u=1}^{k+L_k-1} d_{us} - \sum_{t=1}^{k-1} a_{ts} \Big]^+ r_s ,$$

or

$$\sum_{s=J_k}^{k}\Big[\sum_{u=1}^{k+L_k-2} d_{us} - \sum_{t=1}^{k-1} a_{ts}\Big]^{+} r_s + \sum_{s=1}^{J_k-1}\Big[\sum_{u=1}^{k+L_k-1} d_{us} - \sum_{t=1}^{k-1} a_{ts}\Big]^{+} r_s$$

$$> r'_k \ .$$

But this last inequality implies that the integer pair preceding (L_k, J_k) satisfies (3.4.3). As long as

$$(L_k, J_k) > (1,1) \ ,$$

this contradicts the fact that the Terminating Pair (L_k, J_k) was defined to be the smallest pair satisfying (3.4.3). Consequently, (3.4.21) holds in all instances with the single exception,

$$(L_k, J_k) = (1,1) \ .$$

Returning to the problem at hand, we have

$$K(I) + L_{K(I)} \geq I + 3$$

and from the defintion (3.4.19),

$$K(I) \leq I \ ,$$

so that

$$L_{K(I)} \geq 3 \ .$$

Thus, when $k = K(I)$, the exception case $(L_k, K_k) = (1,1)$ is ruled out and (3.4.21) holds for all j, $1 \leq j \leq K(I)$.

Hence,

$$a_{K(I),j}r_j \geq \Big[\sum_{u=1}^{K(I)+L_{K(I)}-2} d_{uj} - \sum_{t=1}^{K(I)-1} a_{tj}\Big]^{+} r_j$$

$$\geq \Big[\sum_{u=1}^{I+1} d_{uj} - \sum_{t=1}^{K(I)-1} a_{tj}\Big]^{+} r_j$$

$$\geq \Big[\sum_{u=1}^{I+1} d_{uj} - \sum_{t=1}^{K(I)-1} a_{tj}\Big] r_j \ , \ j = 1,\ldots,K(I) \ ,$$

and so we have

$$0 \geq \sum_{u=1}^{I+1} d_{uj} - \sum_{t=1}^{K(I)} a_{tj} \geq \sum_{u=1}^{I+1} d_{uj} - \sum_{t=1}^{I} a_{tj} \ ,$$

$$j = 1,\ldots,K(I) \ .$$

But by its defintion (3.4.19), K(I) satisfies (3.4.15). Consequently, the definition of J(I) insures that

$$\Big[\sum_{u=1}^{I+1} d_{uj} - \sum_{t=1}^{I} a_{tj}\Big] r_j \geq 0 \ , \ j = K(I)+1,\ldots,I \ .$$

Combining this result with the preceding inequality, we see that K(I) also satisfies (3.4.16) and (3.4.17).

We have thus proved that there exists integers satisfying (3.4.15), (3.4.16), and (3.4.17). It now remains to show that (3.4.18) is satisfied by the largest such integer $K_{max}(I)$.

Suppose that there is some pair $(\hat{k},\hat{j})$,

$$K_{max}(I) + 1 \leq \hat{k} \leq I \ , \ 1 \leq \hat{j} \leq K_{max}(I) \ ,$$

for which (3.4.18) is false, i.e.,

$$a_{\hat{k}\hat{j}} r_{\hat{j}} > 0 \ .$$

From (3.4.17) for $j = \hat{j}$,

$$\sum_{u=1}^{I+1} d_{u\hat{j}} - \sum_{t=1}^{K_{max}(I)} a_{t\hat{j}} \leq 0 \ ,$$

so that

$$a_{\hat{k}\hat{j}} > 0 \geq \sum_{u=1}^{I+1} d_{u\hat{j}} - \sum_{t=1}^{K_{max}(I)} a_{t\hat{j}} \geq \sum_{u=1}^{I+1} d_{u\hat{j}} - \sum_{t=1}^{\hat{k}-1} a_{t\hat{j}} \ .$$

Our supposition that $a_{\hat{k}\hat{j}} r_{\hat{j}} > 0$ implies that $r_{\hat{j}} > 0$. Hence, the SCOUR formula (3.4.4) together with definition of the Terminating Pair (3.4.3) insures that

$$\left[\sum_{u=1}^{\hat{k}+L_{\hat{k}}-1} d_{u\hat{j}} - \sum_{t=1}^{\hat{k}-1} a_{t\hat{j}} \right]^{+} \geq a_{\hat{k}\hat{j}} > 0 \ .$$

Combining this with the preceding inequality,

$$\left[\sum_{u=1}^{\hat{k}+L_{\hat{k}}-1} d_{u\hat{j}} - \sum_{t=1}^{\hat{k}-1} a_{t\hat{j}} \right]^{+} = \left[\sum_{u=1}^{\hat{k}+L_{\hat{k}}-1} d_{u\hat{j}} - \sum_{t=1}^{\hat{k}-1} a_{t\hat{j}} \right]$$

$$> \left[\sum_{u=1}^{I+1} d_{u\hat{j}} - \sum_{t=1}^{\hat{k}-1} a_{t\hat{j}} \right] \ ,$$

which implies that

$$\hat{k} + L_{\hat{k}} - 1 > I + 1$$

or

$$\hat{k} + L_{\hat{k}} \geq I + 3 .$$

Moreover, since

$$\hat{k} \leq I ,$$

we must have

$$L_{\hat{k}} \geq 3$$

and we can apply (3.4.21) to obtain

$$a_{\hat{k}j} r_j \geq [\sum_{u=1}^{\hat{k}+L_{\hat{k}}-2} d_{uj} - \sum_{t=1}^{\hat{k}-1} a_{tj}]^{+} r_j$$

$$\geq [\sum_{u=1}^{I+1} d_{uj} - \sum_{t=1}^{\hat{k}-1} a_{tj}] r_j , \quad j = 1,\ldots,\hat{k} ,$$

or

$$0 \geq [\sum_{u=1}^{I+1} d_{uj} - \sum_{t=1}^{\hat{k}} a_{tj}] r_j$$

$$\geq [\sum_{u=1}^{I+1} d_{uj} - \sum_{t=1}^{I} a_{tj}] r_j , \quad j = 1,\ldots,\hat{k} .$$

The integer $\hat{k}$ would then satisfy the requirements (3.4.15), (3.4.16), and (3.4.17). But

$$\hat{k} > K_{max}(I) ,$$

while $K_{max}(I)$ was defined to be the largest such integer. This contradiction implies that no such $(\hat{k},\hat{j})$ exists, and hence (3.4.18) follows, proving the lemma.

Lemma 3.4.4. Let $r' \in \mathcal{R}$ and satisfy (3.4.1). Let (L_k,J_k) be the smallest pair satisfying (3.4.3) for each k, $1 \leq k \leq I < n$. Suppose that the resulting SCOUR components

$$a_{kj} , \quad k = 1,\ldots,I , \quad j = 1,\ldots,n ,$$

satisfy (3.4.5), (3.4.6), (3.4.7), (3.4.8), and (3.4.9). Then there exists a smallest pair (L_{I+1},J_{I+1}) realizing (3.4.3) for $k = I+1$, and the resulting SCOUR components $a_{I+1,j}$, $1 \leq j \leq n$, satisfy (3.4.10), (3.4.11), (3.4.12), (3.4.13), and

$$(3.4.22) \quad \sum_{u=1}^{m} d_{uj} \leq \sum_{t=1}^{m} a_{tj} , \quad m = 1,\ldots,I+1 , \quad j = 1,\ldots,n .$$

Proof: In light of Lemma 3.4.2, all the conclusions are immediate except for (3.4.22). Moreover, because of (3.1.2) and (3.4.8), it suffices to prove,

$$(3.4.23) \quad \sum_{u=1}^{I+1} d_{uj} \leq \sum_{t=1}^{I+1} a_{tj} , \quad j = 1,\ldots,I+1 .$$

For $r_j = 0$, (3.4.23) is immediately obtained from (3.4.4a). Therefore, we assume $r_j > 0$ for the remainder

of the proof.

If $L_{I+1} \geq 2$, then (3.4.21) applies,

$$a_{I+1,j} \geq \Big[\sum_{u=1}^{I+L_{I+1}-1} d_{uj} - \sum_{t=1}^{I} a_{tj} \Big]^{+}$$

$$\geq \sum_{u=1}^{I+1} d_{uj} - \sum_{t=1}^{I} a_{tj} \ ,$$

and (3.4.23) follows at once.

For $L_{I+1} = 1$ and $1 \leq j < J_{I+1}$, (3.4.4b) yields (3.4.23) as follows,

$$a_{I+1,j} = \Big[\sum_{u=1}^{I+1} d_{uj} - \sum_{t=1}^{I} a_{tj} \Big]^{+}$$

$$\geq \sum_{u=1}^{I+1} d_{uj} - \sum_{t=1}^{I} a_{tj} \ .$$

It remains to demonstrate (3.4.23) for $L_{I+1} = 1$ and $J_{I+1} \leq j \leq I + 1$.

Applying (3.4.8), (3.1.2), and (3.4.6) to (3.4.3) for $L_{I+1} = 1$, we obtain

$$r'_{I+1} \leq \sum_{s=1}^{J_{I+1}} \Big[\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} a_{ts} \Big]^{+} r_s$$

(3.4.24)

$$\leq \sum_{s=1}^{I+1} \Big[\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} a_{ts} \Big]^{+} r_s \ .$$

First, we consider the case,

$$(3.4.25) \qquad \sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} a_{ts} \geq 0 \ , \ s = 1,\ldots,I \ .$$

Then (3.4.24) becomes

$$\begin{aligned} r'_{I+1} &\leq \sum_{s=1}^{I+1} \left[\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} a_{ts}\right] r_s \\ &= \sum_{s=1}^{I+1} r_s \sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} \sum_{s=1}^{I+1} a_{ts} r_s \\ &= \sum_{s=1}^{I+1} r_s \sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} \sum_{s=1}^{n} a_{ts} r_s \\ &= \sum_{s=1}^{I+1} r_s \sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} r'_t \\ &\leq r'_{I+1} \ , \end{aligned}$$

where the last inequality derives from (3.4.1) with $M = I + 1$ and $m = 1$. Thus,

$$r'_{I+1} = \sum_{s=1}^{I+1} r_s \sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} r'_t \ ,$$

or

$$\begin{aligned} 0 &= \sum_{s=1}^{I+1} r_s \sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I+1} r'_t \\ &= \sum_{s=1}^{I+1} r_s \left[\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I+1} a_{ts}\right] \ . \end{aligned}$$

But since $L_{I+1} = 1$, we have from (3.4.4b),

$$a_{I+1,s} r_s \leq [\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} a_{ts}]^+ r_s$$

$$= [\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} a_{ts}] r_s \quad , \quad s = 1,\ldots,I+1 \ ,$$

or

$$[\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I+1} a_{ts}] r_s \geq 0 \ , \quad s = 1,\ldots,I+1 \ .$$

These two results imply

$$r_s[\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I+1} a_{ts}] = 0 \ , \quad s = 1,\ldots,I+1 \ ,$$

so that

$$\sum_{u=1}^{I+1} d_{uj} = \sum_{t=1}^{I+1} a_{tj}$$

which is the needed result.

Next, consider the case where (2.4.25) fails to hold, i.e., there exists an $\hat{s}$, $1 \leq \hat{s} \leq I$, such that

$$\sum_{u=1}^{I+1} d_{u\hat{s}} - \sum_{t=1}^{I} a_{t\hat{s}} < 0 \ .$$

Then Lemma 3.4.3 guarantees the existence of an integer K(I) satisfying (3.4.15), (3.4.16), and (3.4.17). As before, let $K_{max}(I)$ denote the largest of such integers.

Then, using (3.4.16) we can expand (3.4.24) as follows,

$$
\begin{aligned}
r'_{I+1} &\leq \sum_{s=1}^{I+1} \Big[\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} a_{ts}\Big]^{+} r_s \\
&= \sum_{s=K_{max}(I)+1}^{I+1} \Big[\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} a_{ts}\Big] r_s \\
&= \sum_{s=K_{max}(I)+1}^{I+1} r_s \sum_{u=1}^{I+1} d_{us} \\
&\quad - \sum_{s=K_{max}(I)+1}^{I+1} r_s \sum_{t=K_{max}(I)+1}^{I} a_{ts} \\
&= \sum_{s=K_{max}(I)+1}^{I+1} r_s \sum_{u=1}^{I+1} d_{us} \\
&\quad - \sum_{t=K_{max}(I)+1}^{I} \sum_{s=K_{max}(I)+1}^{n} a_{ts} r_s \qquad (3.4.27) \\
&= \sum_{s=K_{max}(I)+1}^{I+1} r_s \sum_{u=1}^{I+1} d_{us} \\
&\quad - \sum_{t=K_{max}(I)+1}^{I} \sum_{s=1}^{n} a_{ts} r_s \\
&\quad + \sum_{t=K_{max}(I)+1}^{I} \sum_{s=1}^{K_{max}(I)} a_{ts} r_s \\
&= \sum_{s=K_{max}(I)+1}^{I+1} r_s \sum_{u=1}^{I+1} d_{us} - \sum_{t=K_{max}(I)+1}^{I} r'_t \\
&\leq r'_{I+1} ,
\end{aligned}
$$

where the last equality arises from (3.4.18) of Lemma 3.4.3,

$$a_{ts} = 0 \ , \ t = K_{max}(I)+1,\ldots,I \ , \ s = 1,\ldots,K_{max}(I) \ ,$$

and the last inequality derives from (3.4.1) with $M = I + 1$ and $m = K_{max}(I)+1$.

Two cases must be differentiated at this point, depending on whether $r'_{I+1} = 0$ or $r'_{I+1} > 0$.

First, when $r'_{I+1} = 0$, we have from (3.4.27)

$$0 = \sum_{s=1}^{I+1} [\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} a_{ts}]^{+} r_s$$

which implies for $r_j > 0$,

$$[\sum_{u=1}^{I+1} d_{uj} - \sum_{t=1}^{I} a_{tj}]^{+} = 0 \ ,$$

or

$$\sum_{u=1}^{I+1} d_{uj} \leq \sum_{t=1}^{I} a_{tj} \leq \sum_{t=1}^{I+1} a_{tj} \ ,$$

which is just the needed result.

Next, when $r'_{I+1} > 0$, we first show that the Terminating Pair $(1, J_{I+1})$ must be such that

$$[\sum_{u=1}^{I+1} d_{uJ_{I+1}} - \sum_{t=1}^{I} a_{tJ_{I+1}}]^{+} r_{J_{I+1}} > 0 \ .$$

From (3.4.3), for $k = I+1$,

$$0 < r'_{I+1}$$

$$\leq \sum_{s=J_{I+1}+1}^{I+1} [\sum_{u=1}^{I} d_{us} - \sum_{t=1}^{I} a_{ts}]^{+} r_s$$

$$+ \sum_{s=1}^{J_{I+1}} [\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} a_{ts}]^{+} r_s$$

$$= \sum_{s=1}^{J_{I+1}} [\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} a_{ts}]^{+} r_s \ .$$

For $J_{I+1} = 1$, the result is obvious. On the other hand, for $2 \leq J_{I+1} \leq I + 1$,

$$[\sum_{u=1}^{I+1} d_{uJ_{I+1}} - \sum_{t=1}^{I} a_{tJ_{I+1}}]^{+} r_{J_{I+1}} = 0$$

would imply that $(1, J_{I+1}-1)$ qualifies as the Terminating Pair. Hence, we must have

$$[\sum_{u=1}^{I+1} d_{uJ_{I+1}} - \sum_{t=1}^{I} a_{tJ_{I+1}}]^{+} r_{J_{I+1}} > 0 \ ,$$

as claimed. This result in turn implies both

$$r_{J_{I+1}} > 0$$

and

$$\sum_{u=1}^{I+1} d_{uJ_{I+1}} - \sum_{t=1}^{I} a_{tJ_{I+1}} > 0 \ .$$

This last inequality combined with (3.4.16) guarantees that

$$J_{I+1} > K_{max}(I) ,$$

so that the SCOUR components become

$$a_{I+1,s}r_s = \begin{cases} = [\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} a_{ts}]^+ r_s = 0 , \\ \quad s = 1,\ldots,K_{max}(I) \\ = [\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} a_{ts}] r_s , \\ \quad s = K_{max}(I)+1,\ldots,J_{I+1}-1 \\ \leq [\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} a_{ts}] r_s , \\ \quad s = J_{I+1} \\ = [\sum_{u=1}^{I} d_{us} - \sum_{t=1}^{I} a_{ts}]^+ r_s = 0 , \\ \quad s = J_{I+1}+1,\ldots,I+1 , \end{cases}$$

and we have

$$0 \leq [\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I+1} a_{ts}] r_s , \quad s = K_{max}(I)+1,\ldots,J_{I+1} .$$

On the other hand, since (3.4.16) holds for $K_{max}(I)$ and $J_{I+1} > K_{max}(I)$,

$$0 \leq [\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} a_{ts}]r_s$$

$$= [\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I+1} a_{ts}]r_s \ , \ s = J_{I+1}+1,\ldots,I+1 \ .$$

Now we can write (3.4.27) as

$$r'_{I+1} = \sum_{s=K_{max}(I)+1}^{I+1} [\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I} a_{ts}]r_s \ .$$

Since

$$a_{I+1,s}r_s = 0 \ , \ s = 1,\ldots,K_{max}(I) \ ,$$

we have

$$r'_{I+1} = \sum_{s=1}^{n} a_{I+1,s}r_s = \sum_{s=K_{max}(I)+1}^{I+1} a_{I+1,s}r_s \ .$$

Applying this result to (3.4.27), we obtain

$$0 = \sum_{s=K_{max}(I)+1}^{I+1} [\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I+1} a_{ts}]r_s \ .$$

As shown above, the right hand side of this equation is a sum of non-negative terms. Hence,

$$0 = [\sum_{u=1}^{I+1} d_{us} - \sum_{t=1}^{I+1} a_{ts}]r_s \ , \ s = K_{max}(I)+1,\ldots,I+1 \ .$$

In particular, for j,

$$I+1 \geq j \geq J_{I+1} \geq K_{max}(I)+1 \ ,$$

we have

$$0 = [\sum_{u=1}^{I+1} d_{uj} - \sum_{t=1}^{I+1} a_{tj}]r_j ,$$

and since $r_j > 0$,

$$\sum_{u=1}^{I+1} d_{uj} = \sum_{t=1}^{I+1} a_{tj} ,$$

which is precisely the last result needed to complete the proof of the lemma.

Lemma 3.4.1 shows that the SCOUR components for $k = 1$ exist and satisfy (3.4.5), (3.4.6), (3.4.7), (3.4.8), and (3.4.9) for $I = 1$. Lemma 3.4.4 then completes the finite induction argument and demonstrates that the SCOUR construction determines an $n \times n$ real matrix (a_{kj}) satisfying (2.1.8), (2.1.9), (3.2.1), (3.2.3), and

$$\sum_{t=1}^{m} a_{tj} \leq 1 , \quad m = 1,\ldots,n , \quad j = 1,\ldots,n .$$

Moreover, from (3.2.3) with $m = n$, we have

$$1 \geq \sum_{t=1}^{n} a_{tj} \geq \sum_{u=1}^{n} d_{uj} = 1 , \quad j = 1,\ldots,n ,$$

so that (2.1.7) is also satisfied. Consequently, by Theorem 3.2.1, $(a_{kj}) \in \mathcal{A}((d_{uj}))$. Since (a_{kj}) and r' satisfy the relation (3.2.1), it follows that $r' \in \mathcal{R}((d_{uj}))$ and the theorem is proved.

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13 ABSTRACT

The notion of a promise date policy is introduced into an n-period inventory model. The $j^{\underline{th}}$ period consists of the arrival of replenishment stock x_j, receipt of deterministic demand r_j, followed by the issuance of promise dates to all new orders and then the shipment of available stock against outstanding orders promised for the $j^{\underline{th}}$ or earlier periods. The demand schedule $r = (r_1, r_2, \ldots, r_n)$ is associated with an $n \times n$ matrix (d_{uj}) called an urgency structure, $d_{uj} r_j$ being that portion of demand r_j which requires a promise date p such that $j \leq p \leq u$. An $n \times n \times n$ real array is called a promise date policy if and only if the following conditions are satisfied

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$$\sigma_{puj} \geq 0 \ , \ p = 1,\ldots,n \ , \ u = 1,\ldots,n \ , \ j = 1,\ldots,n \ ,$$

$$\sum_{p=1}^{n} \sigma_{puj} = 1 \ , \ u = j,\ldots,n \ , \ j = 1,\ldots,n \ ,$$

$$\sigma_{puj} = 0 \ , \ p > u \text{ or } p < j \ , \ p = 1,\ldots,n \ ,$$
$$u = 1,\ldots,n \ , \ j = 1,\ldots,n \ .$$

The set of all such arrays is denoted by Ω. A policy $(\sigma_{puj}) \in \Omega$ is said to satisfy the FIFO property if promise dates are assigned to orders in the sequence of their receipt. The set of FIFO policies is denoted by $\Omega_F \subset \Omega$. Each $(\sigma_{puj}) \in \Omega$ determines a vector $r' = (r'_1, r'_2, \ldots, r'_n)$ by the relation,

$$r'_k = \sum_{j=1}^{n} \sum_{u=1}^{n} \sigma_{kuj} d_{uj} r_j \ , \ k = 1,\ldots,n \ .$$

r' is called a deferred demand schedule and its component r'_k is the total amount of demand promised for k^{th} period shipment. The deferred demand schedule r' determines a quantity y_j called the net stock level at the end of the j^{th} period,

$$y_j = \sum_{k=1}^{j} x_k - \sum_{k=1}^{j} r'_k \ .$$

The cost of holding stock $y_j \geq 0$ through the end of the j^{th} period is $h_j(y_j)$, $y_j \geq 0$. When $y_j < 0$, $(-y_j)$ is the total amount of all unshipped orders promised in the j^{th} or earlier periods. The shortage cost in this instance is $h_j(y_j)$, $y_j < 0$. For a fixed replenishment schedule, the total cost is then

$$\sum_{j=1}^{n} h_j(y_j) \ .$$

An optimal promise date policy $(\sigma^*_{pu_j})$ is one which minimizes this cost over all policies in Ω.

The set of all deferred demand schedules corresponding to policies in Ω is denoted by $\mathcal{R}((d_{uj}))$. It is shown that $\mathcal{R}((d_{uj}))$ consists of all n-dimensional vectors r' satisfying the conditions,

$$r'_k \geq 0 \ , \quad k = 1,\ldots,n \ ,$$

$$\sum_{k=1}^{n} r'_k = \sum_{j=1}^{n} r_j \ ,$$

$$\sum_{k=1}^{m} r'_k \leq \sum_{j=1}^{m} r_j \ , \quad m = 1,\ldots,n \ ,$$

$$\sum_{k=m}^{M} r'_k \geq \sum_{j=m}^{M} r_j \sum_{u=1}^{M} d_{uj} \ , \quad 1 \leq m \leq M \leq n \ .$$

The set $\mathcal{R}_F((d_{uj}))$ of deferred demand schedules corresponding to policies in Ω_F is shown to consist of all elements in $\mathcal{R}((d_{uj}))$ satisfying the following additional condition: for each pair of integers (m,M), $1 \leq m \leq M \leq n$, such that

$$\sum_{j=m}^{M} r_j \sum_{u=1}^{M} d_{uj} > 0$$

the following inequality holds,

$$\sum_{k=1}^{M} r'_k \geq \sum_{j=m}^{M} r_j \sum_{u=1}^{M} d_{uj} + \sum_{j=1}^{m-1} r_j \ .$$

The search for optimal policies can then be carried out in the set $\mathcal{R}((d_{uj}))$, or for the FIFO problem, in the set $\mathcal{R}_F((d_{uj}))$. Once an optimal element has been found in $\mathcal{R}((d_{uj}))$ or $\mathcal{R}_F((d_{uj}))$, then appropriate constructions are given for determining a corresponding optimal promise date policy in Ω or Ω_F respectively.

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